THE GROUP OF A TACTICAL CONFIGURATION.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, December 29, 1904.)

1. A tactical configuration, connected with the abelian group G_{p^m} of type $(1, 1, \cdots, 1)_m$ and serving to define the general linear homogeneous group H modulo p on m variables, has been given by Professor Moore.* As an obvious generalization, consider the configurations defining the various subgroups of H. The example that I give had its origin in the following problem:† Required the number N_n of all possible ways of separating the $2^{2n}-1$ operators other than identity of $G_{2^{2n}}$ into 2^n+1 sets each of 2^n-1 operators, such that the operators of any set together with identity form a subgroup G_{2^n} and such that no two sets have a common operator. Here $G_{2^{2n}}$ is assumed to be an abelian group of type $(1, 1, \cdots, 1)_{2n}$; for example, the group of all linear transformations on $2n$ variables which multiply each variable by ± 1.

2. That such a separation is always possible is easily shown. The group of automorphisms of $G_{2^{2n}}$ may be taken concretely as the group of all linear homogeneous transformations modulo 2 on $2n$ variables. The latter contains‡ a transformation S whose characteristic equation of degree $2n$ is irreducible modulo 2, so that S is of period $2^{2n}-1$. In particular, an operator Σ of period 2^n-1 occurs. Let $I, a_1, b_1, a_1b_1, \cdots$ be the operators of a subgroup G_{2^n} of $G_{2^{2n}}$. A table of the operators of the latter may be formed with those of G_{2^n} in the first row and with the operators $I, a_2, b_2, \cdots$ of a second G'_{2^n} as multipliers. We may choose $\Sigma = \Sigma_1\Sigma_2$, where Σ_i permutes cyclically the 2^n-1 elements $a_i, b_i, a_ib_i, \cdots$, written in a suitable order. As the first set S_1 we take $a_1, b_1, a_1b_1, \cdots$; as the second set S_2 we take $a_2, b_2, \cdots$. To form the third set S_3, take any element, as a_1a_2, in neither S_1 nor S_2, and apply to it the powers of Σ; there result $a_1a_2, b_1b_2, \cdots$. To form the i^{th} set take any element not in $S_1, S_2, \cdots, S_{i-1}$ and apply to it the powers of Σ. In this way we obtain 2^n+1 sets with the desired properties.

* BULLETIN, vol. 2 (1895), pp. 33–43.

† For $n=2$, see Burnside, Theory of Groups, p. 60, ex. 2; errata, p. xvi.

‡ Linear Groups, p. 236.

3. For $n = 2$, we take $\Sigma = (a, b, ab)(A, B, AB)$ and obtain

$$S_1 = \{a, b, ab\},\ S_2 = \{A, B, AB\},\ S_3 = \{aA, bB, abAB\},$$
$$S_4 = \{bA, abB, aAB\},\ S_5 = \{abA, aB, bAB\}.$$

A suitably chosen substitution of period 5 on the 15 letters (§2) will permute the S_i in a cycle. The following substitutions leaving a and b unaltered: $(A, aA)(B, bB)$,* (A, B, AB) induce on the S_i the substitutions $(S_2S_3)(S_4S_5)$ and $(S_3S_5S_4)$, respectively. Finally, $(a, b)(A, B)$ induces (S_4S_5). Hence the S_i may be permuted in all 120 ways. Next, in view of their origin, each set S_i is unaltered by Σ and its powers, but by no further substitution. Indeed, if each S_i is unaltered by T, we have $T = T_1T_2$, where T_1 affects the a, b, ab in the same way that T_2 affects A, B, AB. But if the T_i are transpositions, S_4 and S_5 are permuted.

The group of the configuration $S_1, \cdots, S_5$ is an imprimitive G_{360}^{15} which gives rise to all 120 *permutations of the S_i.*

4. For $n = 3$, we take as Σ

$$(a, abc, c, bc, ab, b, ac)(A, ABC, C, BC, AB, B, AC),$$

the first seven elements forming S_1, the second seven forming S_2. The remaining sets are formed as in §2:

$$S_3 = \{aA, bB, abAB, cC, acAC, bcBC, abcABC\},$$
$$S_4 = \{cA, aB, acAB, abC, abcAC, bBC, bcABC\},$$
$$S_5 = \{abA, cB, abcAB, acC, bcAC, aBC, bABC\},$$
$$S_6 = \{acA, abB, bcAB, abcC, bAC, cBC, aABC\},$$
$$S_7 = \{abcA, acB, bAB, bcC, aAC, abBC, cABC\},$$
$$S_8 = \{bA, bcB, cAB, aC, abAC, abcBC, acABC\},$$
$$S_9 = \{bcA, abcB, aAB, bC, cAC, acBC, abABC\}.$$

Thus Σ leaves each S_i unaltered. Any substitution T on the 63 letters which leaves unaltered each S_i is a power of Σ. In fact, T must affect the large letters in the same way that it does the small, in view of S_3. Now $T\Sigma^s$, where s is suitably chosen,

* The further cycles $(AB, abAB)(bA, abA)(aB, abB)(aAB, bAB)$ are suppressed. The shorter notation suffices as it gives the new generators. The same remark applies throughout.

will leave aB, and hence a, B, A, b, unaltered. But cA occurs in S_4, so that c, and hence C, are unaltered. Hence $T\Sigma^r$ is the identity.

Next S_1 may be thrown into any S_i. Further,

$$(A, aA)(B, bB)(c, cC), \qquad (A, ABC, C, BC, AB, B, AC),$$

each leaving a, b, c unaltered, induce on the S_i the substitutions

$$(S_2S_3)(S_4S_6)(S_7S_9)(S_5S_8), \qquad (S_3S_6S_8S_5S_9S_4S_7),$$

respectively. Hence the group induced on the S_i is triply transitive. The following substitution, leaving S_1, S_2, S_3, a, and A unaltered,

$$(b, bc, ac)(c, ab, abc)(B, BC, AC)(C, AB, ABC)$$

induces the substitution $(S_4S_5S_7)(S_6S_8S_9)$. But there is no substitution T corresponding to one leaving fixed S_1, S_2, S_3, and replacing S_4 by S_6, S_8, or S_9. Employing $T\Sigma^r$ instead of T, where r is suitably chosen, we may suppose that a and A are also fixed. Let then T replace S_4 by S_6. Hence T replaces c by ac, and B by ABC, in view of the coefficients of A and a in S_4 and S_6. But S must affect the large letters in the same way that it affects the small, in view of S_3. Hence T replaces C by AC, and b by abc. Hence T replaces abC of S_4 by $bcAC$, not in S_6. Similarly, T cannot replace S_4 by S_8 or S_9. Finally, an induced substitution which leaves S_1, S_2, S_3, S_4 each fixed, leaves every S_i fixed and is the identity.

The group of the configuration $S_1, \cdots, S_9$ is an imprimitive $G^{63}_{1512\cdot 7}$ which gives rise to a triply transitive G^{9}_{1512} on the S_i.

5. Since the $2n$-ary linear homogeneous group H modulo 2 has the order

$$(2^{2n} - 1)(2^{2n} - 2)(2^{2n} - 2^2) \cdots (2^{2n} - 2^{2n-1}),$$

we conclude that $N_2 = 2^3.7$, $N_3 = 2^{12}.3.5.31$. In fact,* every separation of the required kind is conjugate within H with that obtained by Σ in §§3–4.

UNIVERSITY OF CHICAGO,
October, 1904.

* A direct proof is given in the *Amer. Math. Monthly*, Nov., 1904.

APPLICATION OF THE THEORY OF CONTINUOUS GROUPS TO A CERTAIN DIFFERENTIAL EQUATION.

BY MR. J. E. WRIGHT.

(Read before the American Mathematical Society, December 29, 1904.)

THE following question is propounded for solution in "Wolstenholme's Mathematical Problems" (No. 1964):

"Along the normal to a curve at P is measured a fixed length PQ; O is a fixed point, and the curve is such that the circle described about OPQ has a fixed tangent at O. Find the differential equation, the general integral, and the singular solution."

The main interest of the question lies in the fact that, although it was proposed without any reference to the theory of groups, it can be completely solved by straightforward application of that theory, and in addition it furnishes a good illustration of the points of connection between differential equations of the first order and that theory.

Take O for origin, and the fixed tangent at O for axis of x. Since PQ is constant in length, it is clear that all lineal elements ab satisfying the differential equation in question, and having their points on the circle OPQ, meet that circle at a constant angle.

Hence, if there exists a group whose path curves are $(x^2 + y^2)/y = \text{constant}$ and which preserves angles, the differential equation admits that group.

Let $\xi\, \partial f/\partial x + \eta\, \partial f/\partial y$ be the symbol of the infinitesimal transformation of the group. We must have $\xi + i\eta = \phi(x + iy)$, since the transformation is conformal, and the differential equation of the path curves is $\xi dy - \eta dx = 0$.

The differential equation of the curves $(x^2 + y^2)/y = \text{constant}$ is

$$2xydx - (x^2 - y^2)dy = 0.$$

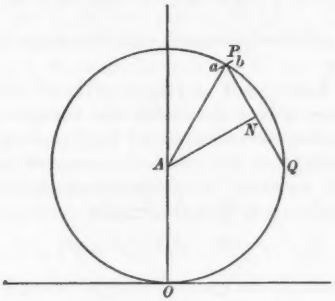
Hence we must have $\xi/(x^2 - y^2) = \eta/2xy$, and these are both equal to $(\xi + i\eta)/(x + iy)^2$. Therefore we take $\xi = x^2 - y^2$ and $\eta = 2xy$, and all the conditions are satisfied.

The finite equations of the group are given by integrating

$$\frac{dx}{x^2 - y^2} = \frac{dy}{2xy} = dt.$$

Of these equations we know the integral of the first, and therefore the remaining integral is given by a quadrature. This second integral is $x/(x^2 + y^2) + t = \text{constant}$.

Pursuing the general method, we take $(x^2 + y^2)/y$, $x/(x^2 + y^2)$, as new variables Y, X, respectively, and we know that the equation will reduce to the form $F(Y, Y') = 0$.



Let $PQ = c$, and let r be the radius of the circle OPQ. Also let p denote dy/dx. AN is obviously parallel to ab, and therefore

$$\tan^{-1}\frac{y - r}{x} = \tan^{-1} p + \tan^{-1}\frac{c}{\sqrt{4r^2 - c^2}}.$$

This gives as the differential equation

$$\frac{c}{\sqrt{\left(\frac{x^2 + y^2}{y}\right)^2 - c^2}} = \frac{(y^2 - x^2) - 2pxy}{p(y^2 - x^2) + 2xy},$$

which reduces to the form

$$x^2 + 2xyp - y^2 = cy\sqrt{1 + p^2}.$$

Now

$$\frac{dY}{dx} = \frac{2xy + p(y^2 - x^2)}{y^2}, \qquad \frac{dX}{dx} = \frac{(y^2 - x^2) - 2pxy}{(x^2 + y^2)^2}.$$

Hence our differential equation becomes

$$\frac{c}{\sqrt{Y^2 - c^2}} = \frac{Y^2}{Y'},$$

the integral of which is

$$\frac{\sqrt{Y^2 - c^2}}{Yc} = X + \text{constant}.$$

If we substitute the old variables, this becomes

$$(x^2 + y^2)(k^2c^2 - 1) + 2c^2kx + c^2 = 0,$$

where k is an arbitrary constant, and this is the required general integral.

Again, we know that if $f(x, y, p) = 0$ admits the group $\xi f_x + \eta f_y$, $f(x, y, \eta/\xi) = 0$ satisfies the equation, and contains as factor any singular solution of the equation. If there are any other factors, they are particular cases of the general integral for certain values of the arbitrary constant.

In our case $f(x, y, \eta/\xi) = 0$ becomes

$$[(x^2 + y^2)^2 - c^2y^2](x^2 + y^2) = 0.$$

Here $x^2 + y^2 = 0$ is the general integral when $k = \infty$, and

$$(x^2 + y^2)^2 - c^2y^2 = 0$$

is the singular solution.

BRYN MAWR, PA.,
October, 1904.

ON THE QUINTIC SCROLL HAVING A TACNODAL OR OSCNODAL CONIC.

BY PROFESSOR VIRGIL SNYDER.

(Read before the American Mathematical Society, October 29, 1904.)

BESIDES the quintic scrolls having three double conics which were discussed in the BULLETIN (volume 9, pages 236–242), other particular types exist. Two of the double conics may become consecutive, forming a tacnodal conic; or all three may become consecutive, forming an oscnodal conic. The necessary

condition that a nodal curve on a scroll be counted as two consecutive nodal curves is that the two generators which intersect at any point on the curve shall lie in a plane containing the tangent to the curve at that point.

1. Let $x = \lambda$, $y = \lambda^2$, $z = 0$, $w = 1$ be a conic c_1. The tangent plane at λ and passing through the point (0, 0, 1, 1) is $\lambda^2(z - w) + 2\lambda x - y = 0$. Consider a second conic c_2, $x = 0$, $z^2 - yw = 0$. This conic cuts c_1 in two points, (0, 0, 0, 1), (0, 1, 0, 0) and its plane passes through the vertex (0, 0, 1, 1). The latter also lies in the common tangent plane of c_1 and c_2 at (0, 0, 0, 1). A tangent plane to c_1 at λ and passing through (0, 0, 1, 1) will cut c_2 in two points $(0, y_1, z_1, w_1)$. If each such point on c_2 be joined with λ by straight lines, a scroll will be defined having c_1 as tacnodal conic. If any point on c_2 be joined to (0, 0, 1, 1) and through the joining line the tangent planes be drawn to c_1, it is seen that c_2 is a double conic. Between c_1, c_2 exists therefore a (2, 2) correspondence having (0, 1, 0, 0) for a self-corresponding point and (0, 0, 0, 1) for double element. The scroll is consequently a unicursal quintic. On performing the algebraic work indicated, the equation is found to be

$$(yw - z^2 - x^2)^2 (z - w) + 2x^2 z (yw - z^2 - x^2) - yz^2x^2 = 0.$$

The simple conic of the surface * lies in the plane $z - w = 0$ and touches c_1 at (0, 1, 0, 0). An arbitrary plane will cut the scroll in a quintic curve having two double points and two tacnodes. A plane containing a tangent to c_1 will contain a quintic having four consecutives nodes. The common tangent plane to c_1 and c_2 at (0, 0, 0, 1) contains a nodal cubic and two generators for nodal tangents.

2. Now let c_1, c_2 touch each other. They may be defined by

$$c_1 : \quad z = 0, \quad y^2 - xw = 0;$$

$$c_2 : \quad x = 0, \quad y^2 - wz = 0.$$

In parametric form c_1 becomes

$$x = \mu^2, \quad y = \mu, \quad z = 0, \quad w = 1.$$

*See Sisam, BULLETIN, volume 10, page 32.

The equations of a line joining μ on c_1 to $(0, y_1, z_1, w_1)$ on c_2 may be written

$$\frac{x - \mu^2 w}{-\mu^2 w_1} = \frac{y - \mu w}{y_1 - \mu w_1} = \frac{z}{z_1},$$

from which

$$\frac{\mu^2 w - x}{\mu^2 z} = \frac{w_1}{z_1}, \quad \frac{\mu y - x}{\mu z} = \frac{y_1}{z_1}.$$

Since $(0, y_1, z_1, w_1)$ lies on c_2, we have

$$\mu^2(y^2 - wz) - \mu \cdot 2xy + x^2 + xz = 0.$$

The equation of a tangent plane to c_1 at μ has the form

$$\mu^2 w - 2\mu y - \kappa z + x = 0.$$

For a general value of κ, the surface is a sextic scroll, having $x = 0$, $z = 0$ as double generator. If the vertex of the cone lies on c_2, the latter is simple. If $\kappa = 2\mu - 1$, the plane $x + z = 0$ is a factor and c_2 is a double conic on the resulting S_5. c_1 is a tacnodal conic.

3. A further specialization can occur, in which all three conics become consecutive. The possibility of this form was pointed out by Wiman,* who incidentally mentions it while describing scrolls having double conics, but does not discuss it nor derive its equation. Wiman employs the Noether point depiction of the lines of a linear complex into the points of space, projectively equivalent to the system which Lie investigated. The exceptional cases and fundamental configurations are treated in detail in my Göttingen dissertation.†

Let

$$(1) \qquad x_1 x + y_1 y + w_1 w = 0, \qquad y_1 x + z_1 y + w_1 z = 0.$$

define a relation between the points of one space and the lines of another. To the points (x_1, y_1, z_1, w_1), or briefly (x_1), correspond the lines of the complex

$$c^1 : \qquad p_{14} + p_{23} = 0,$$

* Klassifikation af regelytorna af sjette graden, Lund (dissertation), 1892; page 87.

† Ueber die linearen Complexe der Lie'schen Kugelgeometrie.

and to the points of space (x) correspond the lines of the quadratic complex

$$c^2: \qquad p_{34}p_{14} - p_{42}^2 = 0,$$

which consists of all the lines which intersect the conic $w_1 = 0$, $y_1^2 = x_1 z_1$.

To a ruled surface belonging to c^1 corresponds a curve in (x_1); to its double curve corresponds a ruled surface of bisecants, all of whose generators belong to c^2. The line $x = 0$, $y = 0$ is the fundamental line of (x) and the plane $w_1 = 0$ is fundamental in (x_1). To all points in $w_1 = 0$ corresponds the same line $x = 0$, $y = 0$. An important relation exists between the arrangement of points in (x) and the lines of c^2 in (x_1). If a point (x_1, y_1, z_1) be given on $y_1^2 - x_1 z_1 = 0$, $w_1 = 0$, then a plane $x_1 x + y_1 y = 0$ is defined in (x). All the lines of the pencil belonging to c^1 and lying in the plane $x_1 x + y_1 y = 0$ go over into the same point of the conic. All the points of one of these lines go over into lines of a pencil whose vertex is on the conic and which contains the tangent to the conic; hence all the points in the plane correspond to the pencils of lines of c^2 lying in the pencil of planes through a common tangent to the conic.

Given then a scroll S_m of order m, in (x_1), having the fundamental conic for a ρ-fold curve, in its plane lie $m - 2\rho$ generators; hence to S_m corresponds a curve $c_{m-\rho}$ of order $m - \rho$ in (x) having $m - 2\rho$ points on the fundamental line $x = 0$, $y = 0$. To apply this depiction to an S_5 having three double conics, regard one of the conics as the fundamental conic of c^2 and transform the surface into point space. The generators will go into the points of a twisted cubic, cutting $x = 0$, $y = 0$ once, and the residual double conic will become two hyperboloids, each having two generators belonging to c^1 in its second system. If one of these lines be chosen as $x = 0$, $y = 0$ and the system be depicted back into (x_1), the double conic will coincide with the fundamental conic counting as the tacnodal conic on S_5. This surface is then equivalent to that given before. If now the ordinary double conic be taken as fundamental conic, the surface transformed into points of (x), and the same transformation of the fundamental line made as in the preceding case, then the S_5 which is the transform of the last system will have the fundamental conic as osculnodal curve.

From the equations of a generator we derive $y - \lambda x = z\zeta$,

$\zeta(\lambda w - x) = \lambda x$, $\lambda^2(\zeta - 1) = \zeta^2$. By letting $\lambda = (l^2 + 1)/l$, $\zeta = l^2 + 1$, the equations of a generator may be written

$$l^2 z_1 - l w_1 + z_1 - y_1 = 0,$$

$$l^2 y_1 + l^2 w_1 + l(y_1 - x_1) + w_1 = 0.$$

Making these equations identical with (1), we obtain

$$\sigma x = -l, \quad \sigma y = l(l^2 + 1), \quad \sigma z = -l^2, \quad \sigma w = l^2 + 1,$$

as the equations of c_3, the (x) image of the S_5. The curve c_3 lies on the hyperboloid $zw - xy = 0$, which is to be transformed so that $x = 0$, $y = 0$ lies on the surface and cuts c_3 once, all the generators cutting $x = 0$, $y = 0$ shall belong to c^1, but the points of the surface shall not go over into lines of a special linear complex in (x_1). The most general transformation of this type may be defined by

$$y = y', \quad x = cy' - dz', \quad z = x', \quad w = ax' + by' + dw',$$

from which

$$x' = -l^2, \quad y' = l(l^2 + 1), \quad z' = al(l^2 + 1) + bl,$$

$$w' = b(l^2 + 1) + cl^2 + dl(l^2 + 1).$$

The values are to be substituted in (1) for x, y, z, w. The l eliminant in (x_1) has the form

$$\begin{vmatrix} y + dw & cw - x & y + dw & bw & 0 \\ 0 & y + dw & cw - x & y + dw & bw \\ z + aw & -y & z + cw & 0 & 0 \\ 0 & z + aw & -y & z + cw & 0 \\ 0 & 0 & z + aw & -y & z + cw \end{vmatrix} = 0,$$

after a few obvious changes are made in the constants.

A plane through any tangent to the osculating conic will cut from the surface a quintic having six consecutive double points. A plane containing two generators contains a nodal cubic having the generators as nodal tangents.

CORNELL UNIVERSITY,
October 18, 1904.

ON THE DEFORMATION OF SURFACES OF TRANSLATION.

BY DR. BURKE SMITH.

(Read before the American Mathematical Society, December 29, 1904.)

IT is here proposed to find all surfaces of translation which may be deformed so that their generating lines remain generating lines throughout the deformation. The generating lines, $u = \text{const.}$, $v = \text{const.}$, of a surface of translation

$$(1)\quad x = f_1(u) + \phi_1(v),\quad y = f_2(u) + \phi_2(v),\quad z = f_3(u) + \phi_3(v)$$

form a conjugate system, since x, y, z satisfy an equation of the form *

$$(2)\qquad \frac{\partial^2\theta}{\partial u\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}.$$

The square of the lineal element of (1) is,

$$(3)\qquad ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

where

$$E = \Sigma f_i'^2,\quad F = \Sigma f_i'\phi_i',\quad G = \Sigma\phi_i'^2 \qquad (i = 1, 2, 3)$$

and the primes denote derivatives of u and v respectively.

Forming the symbols of Christoffel from (3) we have

$$(4)\qquad \left\{\begin{matrix}1 & 2\\ 1\end{matrix}\right\} = 0 \quad \text{and} \quad \left\{\begin{matrix}1 & 2\\ 2\end{matrix}\right\} = 0.$$

Conversely, if a surface S is referred to a conjugate system of lines and if (4) is true, then S must be a surface of translation, since its coördinates x, y, z must satisfy

$$\frac{\partial^2\theta}{\partial u\partial v} = \left\{\begin{matrix}1 & 2\\ 1\end{matrix}\right\}\frac{\partial\theta}{\partial u} + \left\{\begin{matrix}1 & 2\\ 2\end{matrix}\right\}\frac{\partial\theta}{\partial v},$$

the integration of which gives (1) by virtue of (4). Hence we have the following

* Bianchi-Lukat: Vorlesungen über Differentialgeometrie, p. 110.

THEOREM. *A surface of translation, when referred to its conjugate system of generating lines, is characterized by the equations* (4).

Consider now the images on the Gauss sphere of a conjugate system of lines on any surface.

The relations between the symbols of Christoffel formed for a surface and for the corresponding Gauss sphere are as follows: *

$$(5)\quad \left\{\begin{matrix}1 & 2\\ 1\end{matrix}\right\} = -\frac{D''}{D}\left\{\begin{matrix}1 & 1\\ 2\end{matrix}\right\}'; \quad \left\{\begin{matrix}1 & 2\\ 2\end{matrix}\right\} = -\frac{D}{D''}\left\{\begin{matrix}2 & 2\\ 1\end{matrix}\right\}',$$

where the primes refer to the Gauss sphere and the second fundamental magnitudes of the surface are D, 0, D''.

Suppose now that $D \neq 0$ and $D'' \neq 0$. Then from (5), if $\{^{12}_{1}\} = 0$ and $\{^{12}_{2}\} = 0$ we must also have,

$$(6)\qquad \left\{\begin{matrix}1 & 1\\ 2\end{matrix}\right\}' = 0 \quad \text{and} \quad \left\{\begin{matrix}2 & 2\\ 1\end{matrix}\right\}' = 0$$

and conversely. It follows † that the images on the Gauss sphere of the generating lines of a non-developable surface of translation are geodesics, and we may state the

THEOREM: *The non-developable surfaces of translation are characterized by the fact that when they are referred to their generating lines, these lines have as images on the Gauss sphere the geodesics of the sphere.*

When either $D = 0$ or $D'' = 0$, the surface is a developable surface. In the case of surfaces of translation, if $D'' = 0$ the equations of Codazzi require that $\partial D/\partial v = 0$ and $\{^{22}_{1}\} = 0$. Hence, calculating D and D'' from (1), we must have in this case

$$f_1'(\phi_3'\phi_2'' - \phi_2'\phi_3'') + f_2'(\phi_1'\phi_3'' - \phi_3'\phi_1'') + f_3'(\phi_2'\phi_1'' - \phi_1'\phi_2'') = 0,$$

$$\phi_1'(f_2''f_3' - f_3''f_2') + \phi_2'(f_1'f_3'' - f_3'f_1'') + \phi_3'(f_2'f_1'' - f_1'f_2'')$$
$$= U\sqrt{EG - F^2}, \qquad \left\{\begin{matrix}2 & 2\\ 1\end{matrix}\right\} = 0,$$

where U is a function of u alone and the primes denote derivatives. These equations are satisfied when

* Bianchi-Lukat, l. c., p. 135.
† Ibid., l. c., p. 154.

$$\phi_3'\phi_2'' - \phi_2'\phi_3'' = \phi_1'\phi_3'' - \phi_3'\phi_1'' = \phi_2'\phi_1'' - \phi_1'\phi_2'' = 0,$$

which leads to the cylindrical surfaces

$$x = f_1(u) + c_1 v, \quad y = f_2(u) + c_2 v, \quad z = f_3(u) + c_3 v,$$

where c_i = constant.

Bianchi has shown * that surfaces S which may be deformed with preservation of a conjugate system are the associates of surfaces S_0 whose total curvature, in terms of the parameters of their asymptotic lines is of the form, $K = -[\phi(u) + \psi(v)]^{-2}$. We now proceed to determine the surfaces S_0 which are the associates of the surfaces of translation. Since a surface referred to a conjugate system of lines has the same spherical representation as its associate surface referred to its asymptotic lines, (5) may be interpreted not only for the surfaces of translation, but also for the surfaces to which they are associated. The following relations exist between the symbols of Christoffel formed for any surface referred to its asymptotic lines and for the corresponding Gauss sphere: †

$$(7) \qquad \left\{ \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \right\} = -\left\{ \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \right\}', \quad \left\{ \begin{matrix} 2 & 2 \\ 1 & \end{matrix} \right\} = -\left\{ \begin{matrix} 2 & 2 \\ 1 & \end{matrix} \right\}'.$$

Then by (5), (6) and (7) the surfaces to which the surfaces of translation are associated are such that

$$(8) \qquad \left\{ \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \right\} = 0 \quad \text{and} \quad \left\{ \begin{matrix} 2 & 2 \\ 1 & \end{matrix} \right\} = 0.$$

But (8) is the necessary and sufficient condition that the parametric lines should be geodesics, and hence the surface S_0 to which a surface of translation is associated is one on which both systems of parametric lines are at the same time asymptotic lines and geodesic lines. Since such lines are necessarily straight lines, the surface S_0 must be a doubly ruled surface, the two systems of parametric lines being the rectilinear generatrices of the surface. But since the only surface possessing two systems of straight lines lying wholly on the surface is a quadric, we may state the

THEOREM. *The surfaces to which the non-developable surfaces*

* Bianchi-Lukat, p. 337.
† Ibid., l. c., p. 127.

of translation are associated are the quadric surfaces, and the rectilinear generatrices of the quadric surfaces correspond to the generating lines of the surfaces of translation. The non-developable surfaces of translation which may be deformed with preservation of generating lines are those surfaces, and only those, which are the associates of the quadric surfaces whose curvature, when expressed in terms of their asymptotic lines, is of the form

$$K = -[\phi(u) + \psi(v)]^{-2}. \tag{9}$$

The quadrics whose curvature is of the above form will now be determined. Consider first a central quadric. The equations of a central quadric in terms of the parameters of the asymptotic lines are

$$x = a\left(\frac{1+uv}{u+v}\right), \quad iy = b\left(\frac{1-uv}{u+v}\right), \quad z = c\left(\frac{u-v}{u+v}\right). \tag{10}$$

By direct calculation from (10) we have for the total curvature

$$K = \frac{a^2b^2c^2(u+v)^4}{[a^2b^2(u-v)^2 - a^2c^2(1-uv)^2 + b^2c^2(1+uv)^2]^2}.$$

The only case in which K is of the required form is when $b^2 = c^2 = a^2$, and the quadric then reduces to a sphere, for which $K = 1/a^2$.

Consider next the paraboloids. The equations of the paraboloids in terms of the parameters of the asymptotic lines are

$$x = \frac{u+v}{2\sqrt{l}}, \quad y = \frac{u-v}{2i\sqrt{m}}, \quad 2z = uv, \tag{11}$$

where l and m are constants.

From (11) we have

$$K = \frac{16lm}{[4 + (l-m)u^2 + 2(l+m)uv + (l-m)v^2]^2},$$

and hence K is here of the required form when and only when $l = -m$, in which case (11) are the equations of the equilateral hyperbolic paraboloid.

Thus the only quadric surfaces whose curvature expressed in terms of the parameters of the asymptotic lines is of the form (9) are the sphere and the equilateral hyperbolic para-

boloid. Now the associate surfaces of a sphere are the minimum surfaces,* and the associate surfaces of the equilateral hyperbolic paraboloid are the surfaces of translation whose generating lines are plane and lie in perpendicular planes.† Hence we may state the

THEOREM: *The only non-developable surfaces of translation which may be deformed so that their generating lines remain generating lines throughout the deformation are the minimum surfaces and those surfaces of translation whose two systems of generating lines are plane and lie in perpendicular planes.*

PURDUE UNIVERSITY,
October, 1904.

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS AT HEIDELBERG.

THE International Congress of 1904 was held at Heidelberg, August 8 to 13, under the efficient management of the Deutsche Mathematiker-Vereinigung. The attendance, 336, showed continued advance in comparison with the preceding congresses of Zürich and Paris, the proceedings and papers were of much interest and in some cases of notable importance, and the local arrangements, both for meetings and for hospitable welcome, were generous and admirable. One could have wished only for an increase of the day's normal twenty-four hours, or a distribution of sectional meetings which would render it less difficult to attend more than one section. The programme for the week was in its main outlines as follows:

Monday, August 8.— Informal gathering at the City Hall, with an address of welcome by Professor Moritz Cantor.

Tuesday, August 9.— General session. Addresses of welcome by the son of the Grand Duke of Baden and other official representatives. Address in commemoration of Jacobi by Professor Königsberger. Organization of the six sections. Banquet in the City Hall.

Wednesday, August 10.— Meetings of the sections. Reception at the grand ducal palace at Schwetzingen.

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* See Eisenhart, BULLETIN, March, 1902, p. 243.
† Bianchi-Lukat, p. 338.

Thursday, August 11. — General session. Presentation of the History of the Deutsche Mathematiker-Vereinigung by Professor Gutzmer. Presentation of Volume I of the Encyclopedia by Professor Klein. Addresses by Professors Greenhill and Painlevé. Opening of the exhibitions of models, apparatus, and mathematical literature of the last decade. Neckar excursion.

Friday, August 12. — Meetings of the sections. Demonstrations of models, etc. Reunion and "Kommers" in the castle restaurant.

Saturday, August 13. — Final general session. Resolutions. Addresses by Professors Segre and Wirtinger.

In his preliminary address of Monday evening Professor Cantor, of Heidelberg, as "Deutscher in Deutschland, Badenser in Baden, Heidelberger in Heidelberg," extended to the gathering from many lands a graceful welcome. Zürich, the home of the Bernoullis, of Euler, of Steiner, had offered the first international congress the charms of its renowned lake and of a noble city. The second had met in an exposition year at Paris. With the attractions of these cities Heidelberg might not compete. But a congress consists not merely of sights and beauties of nature, but rather of two quite definite things — sessions and non-sessions, the latter lasting much longer than the former. That these non-sessions might result in the greatest general satisfaction was the most earnest wish of Heidelberg and of himself.

On Tuesday morning Professor Heinrich Weber, of Strassburg, delivered his opening address as president of the congress. He called to mind with brief words of characterization some of the eminent mathematicians who have passed away since the first congress — Weierstrass, Hermite, Sylvester, Salmon, Lie, Brioschi, Cremona, Christoffel, Fuchs — and referred to certain general lines of present mathematical progress.

The son and heir of the Grand Duke of Baden was received with enthusiasm and made an excellent address as honorary president. Official addresses of welcome were also given on behalf of the ministry, the universities of Baden, and the city of Heidelberg.

The commemoration of the hundredth anniversary of the birth of Jacobi in this general session was marked by the delivery of an extended biographical address and the presentation

of a stately biography,* both by Professor Leo Königsberger. The presence of the daughters of Jacobi lent an additional element of personal interest.

The six sections organized on Tuesday afternoon, with brief addresses by the respective "Einführende," and the general introductory events culminated in the evening with the grand banquet in the city hall, attended by some 600 members and guests. The speakers—alternating with the courses—included, besides the two presidents of the congress, Professor Klein, of Göttingen, who expressed the thanks of the Deutsche Mathematiker-Vereinigung to the authorities who had coöperated with it; Professor Krazer, of Karlsruhe, in honor of the foreign guests and Professor Geiser, of Zürich, in response; Professor Noether, of Erlangen, in honor of Heidelberg, and Professor Heffter, of Bonn, in honor of the ladies.

The visit to the grand ducal palace at Schwetzingen was the social event of Wednesday and indeed of the entire week. The visitors were met at the station by assembled fire companies, veterans, musical societies and school children, and had the distinction — unusual for mathematicians — of marching to the castle gates between these ranks of honor. The grand duke himself was in the distant mountains, but his son and heir represented him a[illegible]t not less admirably than on preceding days as honorary president of the congress. Strolls in the pleasant old park and a collation in the orangery brought this Nicht-Sitzung to a successful end.

Thursday morning the congress held its second general session. Professor Gutzmer, of Jena, secretary of the Deutsche Mathematiker-Vereinigung, presented to the congress a copy of his admirable new history of the Vereinigung. He referred particularly in his brief presentation address to the questions which can only be properly settled by coöperation of mathematicians, for example, questions of mathematical education, of the applications of mathematics, of the preparation of reports, encyclopedias, etc. To these questions the Vereini-

* By arrangement with the publishing house of Teubner, copies of this biography were supplied to members of the congress at the nominal price of five marks. The addresses of Professors Königsberger, Greenhill, Painlevé, and Wirtinger, as well as Professor Gutzmer's History, were distributed gratis, a great convenience to those who did not readily understand the spoken language. Here should be mentioned also the very useful Tageblatt, which published the daily programme, lists of those in attendance and their local addresses, excursions, notices, etc.

gung had devoted particular attention and it was therefore fitting to review its activity and progress in Heidelberg, where fifteen years earlier (September, 1889) Georg Cantor gave a new impulse toward the foundation of a society of mathematicians. The history is dedicated to the participants of the Heidelberg congress, each of whom received a copy.

In the absence of Professor von Dyck, chairman of the academic commission, Professor Klein, of Göttingen, presented to the congress the now completed Volume I of the Encyclopedia of the Mathematical Sciences. The final installment of the volume contains a general introductory report by the chairman, a special preface by the editor, Professor Meyer, and, among other matters, a detailed author and subject index. Thanks were extended to all whose generous coöperation has rendered the success of the work possible; to Professor Meyer, who gave the first impulse to the plan and who now has the satisfaction of seeing its partial accomplishment; finally to the publishing house which has placed its great resources without limitation at the service of the undertaking. Volumes II–VI are progressing at such a rate that their completion can be foreseen in a reasonable period; but at the same time the scope of the work becomes ever wider. Professor Molk would present the first part of a French edition prepared by the coöperation of the most eminent French authors. This may be regarded as a second revised edition of the original, and it may be hoped that later a new German edition will follow which shall be still more complete.

Professor Molk, of Nancy, followed in explanation of the French edition. It is not a mere translation, although the general character of the German edition is preserved. French mathematicians and engineers have edited the articles of the German edition with due reference to French usage, additions being marked by asterisks. In each case there has been exchange of views between the German and the French writer —offering thus the first example of collaboration between numerous mathematicians belonging to different countries and speaking different languages. The first part, now presented to the international congress, contains:

1. An article on the foundations of arithmetic, based on the German of H. Schubert, by J. Tannery and J. Molk.

2. An article on combinatory analysis and on determinants, based on the German of E. Netto, by H. Vogt.

3. The first part of an article on irrational numbers, based on the German of A. Pringsheim, by J. Molk.

The French edition is published by Teubner and by Gauthier-Villars.

Professor Painlevé of Paris, delivered an address on "The modern problem of the integration of differential equations."

He referred to the origin of the calculus in connection with the scientific study of natural phenomena, and to the direction of the first applications of this study, differentiation seeking always to analyze the phenomenon, integration always to reconstitute it. The successors of Newton and Leibniz, limiting themselves to the simplest facts in each class, achieved brilliant results. Every advance in analysis had its immediate echo in physics, and conversely. It was the most glorious and fruitful epoch in the history of mathematics, which seemed to be in truth the key of the universe.

The following period — of Cauchy and others — was occupied with efforts to enlarge the bounds of integrability and appears at first view barren in comparison. The apparent deflection of mathematics from the real to the imaginary seems arbitrary and unfruitful, but only to a superficial critic, as is illustrated by the significance of the theory of functions of the complex variable for the Taylor expansion of a function of a real variable.

Complex variables once introduced, the integration of a differential equation could be attempted in two different ways: (1) one could seek to reduce it to simple equations (quadratures, linear equations, etc.), that is to aim — in accordance with the traditions of the older analysis — at the formal integration of the equation; (2) failing a reduction of this sort, one could seek to study the general integral by approximations, determining its properties, singularities, etc., in the entire complex plane.

In the end the integration of a differential equation will consist in two successive operations: (1) reduction of the equation to irreducible equations; (2) direct study of the integrals of these irreducible equations in the entire complex plane.

In order to define reducibility, let us consider an algebraic differential equation, for example,

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

The general integral $y(x)$ of this equation depends on an arbitrary constant u; in other words, the equation defines a function y of two variables x and u, or rather an infinite number of such functions (which may be derived from any one by replacing u by an arbitrary function of u). It is admissible also to consider u as a function of x and y; this function satisfies the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} f = 0, \tag{2}$$

which is equivalent to the preceding equation. The general integral of equation (1) will be called reducible if one can adjoin to equation (2) other (algebraic) equations in x, y, u, $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$, $\cdots$, which are consistent with (2) without being derived from it. The definition may be extended to an equation — or a differential system — of any order.

On this definition can be based a proposition which dominates the whole theory of reducibility, namely, in the case of the first order:

When a differential equation is reducible, among the reduced systems there is always one, of lowest differential order, which has the following properties: (1) It is automorphic, that is, its general solution $u(x, y)$ is deduced from any particular solution $u_1(x, y)$ by the transformations $u = \psi(u_1)$ of a group; (2) all the other reduced systems are derived from this by replacing u by a function $U(u)$ which satisfies an algebraic differential equation (in u and U) chosen arbitrarily.

This theorem, which may be extended immediately to a differential equation of any order, shows the fundamental reason for the great and almost exclusive share which the theory of groups has in the reduction of differential equations.

The simple enumeration of continuous groups for one variable leads to the theorem:

When an equation of the first order is reducible, four cases only can present themselves: (1) The equation is integrable algebraically; (2) it admits an algebraic integrating factor; (3) it admits an integrating factor whose logarithm has its two first derivatives algebraic; (4) a first integral $u(x, y)$ is given by a differential system of which the general solution is of the form $u = (au_1 + b)/(cu_1 + d)$ where a, b, c and d are arbitrary constants, a system leading back, as is well known, to an equation of Riccati.

In general an equation known to be irreducible must be considered as integrated if one can by any process of indefinite approximation — series, continued fractions, etc. — represent the general integral in its entire domain (real and complex) with an error as small as one pleases and of known limit, such representation exhibiting the fundamental properties of the integral, accounting for initial conditions, etc.

It is natural to seek in the first place the differential equations whose general integrals can be obtained by equating uniform functions to zero, and especially the differential equations which possess uniform first integrals. It must not be overlooked that the search for such first integrals is a problem of extreme difficulty. But in a particular (yet somewhat extended) case the problem can now be attacked with success, viz., when the general integral is itself a uniform function in its entire domain of existence. The problem thus particularized is a direct generalization of the discoveries of Abel and Jacobi. If we limit ourselves to the first order, the problem leads only to known results. But it is quite otherwise with equations of higher order. These may define new transcendentals not expressible by any combinations of the classical transcendentals.

Again by generalizing the last problem a little, we may study differential equations of which the integral is a function with n branches or has fixed critical points. But whatever progress may be achieved in these difficult problems, and in other vaster problems of the same nature, it is evident on the one hand that we shall never exhaust all the perfectly integrable equations, and on the other hand that an equation taken at random will not come under these special types. The interest of these remarkable classes of equations, which is evident from a purely mathematical point of view, might then appear smaller from the point of view of the applications, if we did not know what benefit general methods always derive from precise and difficult problems, exhaustively studied.

When a differential equation which occurs in any problem admits neither formal nor analytic integration, the sole resource is to attack it directly by the aid of processes of approximation, guiding one's self as much as possible by the real problem represented. It should not be forgotten that the methods of approximation applicable today to any differential equation, if still theoretically imperfect, prove in many cases amply efficacious. Thus the quantitative problem of n bodies

is not yet rigorously solved, but the ephemerides of our solar system are nevertheless calculated for three centuries with surprising precision.

Formal integration, analytic integration as perfect as possible in the complex field, approximate integration in the real domain, these are the three directions of development thus far. At the center of all these stands the theory of functions in a directing and commanding position. It need not be inferred that this dominance will always belong to it. It may one day be judged as is now, for example, the arithmetical work of Gauss, one of the most harmonious and best constructed parts of the mathematical edifice, but a monument of the past. We shall then, perhaps, possess methods more profound and powerful, which will permit us to attack differential equations boldly, concerning ourselves only with the real problem which they represent. But these fruitful and living methods will have sprung from the theory of functions.

Professor Greenhill, of Woolwich, next made an address on "The mathematical theory of the top considered historically."

Beginning with a citation of Klein-Sommerfeld: Theorie des Kreisels, he traced the subject backward through the work of Routh, Hess, Lottner, Jacobi, Poisson, Lagrange, to Euler and Segner, of Halle, taking especially the latter's Specimen theoriæ turbinum (1755) as the effective starting point. Segner was inspired originally by Serson's artificial horizon, a spinning top to be employed in navigation.

An apparatus was exhibited to show gyroscopic motion by a bicycle wheel suspended from a beam by a brass rod and a bicycle hub. Attention was called to the simplicity of the apparatus, which can easily be duplicated. The wheel is spun by a stick and then so deflected as to have a gyroscopic motion, undulating, cusped, or looped, as may be desired.

The fundamental equations of motion were derived by simple geometric consideration of a diagram, on which the conditions of steady motion and slight nutation were also interpreted.

Jacobi's theorem that top motion is compounded of two associated Poinsot movements was analyzed, and the Poinsot rolling cone, as well as the Kirchhoff kinetic analogue and Darboux's deformable hyperboloid, were shown by simple models easily constructed.

A challenge was thrown down to any writers on the theta

function who might be present, questioning its dynamical utility, and suggesting as an alternative the function which is the quotient of two theta functions having a constant phase difference in the argument.

The fine large hall of the university museum used for the opening general session on Tuesday was afterwards devoted to an extensive exhibition of mathematical literature, apparatus, and models of the last decade. This exhibition was formally opened with addresses on Thursday afternoon, and thereafter proved a constant center of active interest.

In opening the exhibition of mathematical literature, Professor Gutzmer referred to the fact that this exhibit is the first of its kind. It is confined to scientific mathematical literature, but even here completeness is not attainable. After presenting statistics of the literary production of the world, both in general and in mathematics, Professor Gutzmer recommended restriction of the scope of future exhibitions of literature, for example to a complete exhibit of the mathematical literature of the country in which the congress is meeting. He referred also to the value for the higher mathematical instruction of a special exhibit of English and American text-books.

Besides periodicals the exhibit included about 900 publications, Teubner contributing about 250, Gauthier-Villars 175. The number in English was small.

The collection of models numbered about 300, including not only the usual styles in plaster, paper, thread and metal, but also integrators, harmonic analyzers, calculating machines, instruments for drawing curves and solving equations, tops, gyroscopes, kinematical models, optical instruments, etc. There were about 25 exhibitors, but the most important collections were shown by Professor Wiener of Darmstadt, Schilling of Halle, the technical schools in Darmstadt and Karlsruhe, Coradi of Zürich, the University of Göttingen, Zeiss of Jena, and Chateau Frères of Paris.

Leibniz's calculating machine was exhibited and its manner of operation explained by Professor Runge of Hannover. It was pointed out that, while the machine probably never worked with entire accuracy, this was due to faulty mechanical construction alone, the principles underlying its construction being the same as in some successful modern machines.

The Wiener exhibit included many models of space curves

and developable surfaces, movable wire models of surfaces of the second order, thread models of intersecting surfaces, etc. In the great variety of Schilling's models the kinematical were notable. Coradi presented beautifully constructed instruments, including an apparatus for drawing parabolas, a harmonic analyzer and an integrator. The Göttingen exhibit included interesting tops.

Zeiss showed a great variety of optical apparatus and in particular a new form of projecting lantern, in which, if the object placed in the apparatus is transparent or translucent, projection is effected by transmitted light in the ordinary manner, while if opaque objects are to be projected, their images are thrown on the screen by reflected light.

Other objects of interest were Christian Wiener's original model of a surface of the third order with twenty-seven straight lines, Brückner's large collection of polyhedrons having congruent faces and congruent or symmetric angles, and Finsterwalder's movable models of minimum surfaces. Several lectures were held in connection with the exhibit.

Thursday evening was devoted to a delightful Neckarfahrt. After an invasion of a Gartenwirthschaft in numbers for whom the landlord unfortunately could not duplicate the miracle of the loaves and fishes, decorated stone-boats were taken in the evening for the romantic descent of the Neckar. Below the arches of the Carlsbrücke, each pier poured a fountain of fire into the river, while high above the town the castle burst into dazzling light with red fires, burning steadily for some fifteen minutes. In the midst of a beautiful display of fireworks from a boat on the river, the pythagorean diagram stood out brilliantly against the sky, an appropriate symbol of the nature of the meeting. The whole effect was finely spectacular.

After the section meetings of Friday there was again a social evening, and again an illumination of the castle, viewed this time from the fine Scheffel terrace. The large open hall of the castle restaurant was crowded for the Kommers which fittingly ended the general festivities of the week. Professor Schubert, of Hamburg, with a gallant and dashing young corps officer on each side, presided with the skill of experience and the dignified enthusiasm of renewed youth. Oratory and singing vied in their contrast with the more serious pursuits of the week.

At the final general session on Saturday the following resolution was adopted, as presented by P. Tannery, A. von Braunmühl, E. Lampe, G. Loria, M. Simon, D. E. Smith, P. Stäckel, E. Wölffing:

"Whereas, the history of mathematics forms to-day a subject of indisputable importance, its value as well from the purely mathematical standpoint as from the pedagogical becomes continually more conspicuous, and it is therefore indispensable to assign it a fitting place in public education,

"Resolved, that the Third International Congress of Mathematicians at Heidelberg (August, 1904), referring to the recommendations of Section 5 of the International Congress for Comparative Historical Investigation (Paris, July, 1900) and of Section 8 of the International Congress of Historians (Rome, 1903), adopts as its own the following recommendation, international in character, as expressed by the congress at Rome:

"(1) That the history of the exact sciences should be taught by the universities, with provision for lectures in the four divisions, mathematics and astronomy, physics and chemistry, natural sciences, medicine.

"(2) That the elements of the history of the sciences should be included in the secondary school program of the several branches of science."

The following resolution was presented by F. Morley and A. Vassilief:

"The Third International Congress of Mathematicians, considering that the publication of the complete works of Euler has great scientific importance, supports the recommendation made to the Carnegie Institution by the mathematical committee of which Professor Moore was chairman, and expresses the hope of its early realization. Considering, on the other hand, that the success of this understanding requires the coöperation of scholars from all lands, whose assemblage for the elaboration of the plan and the discussion of other related questions will be possible during the next congress,

"Resolved, that the Third Congress requests the committee of organization of the next congress to present to it a report on the status of the question as well as on the steps which the congress could take to contribute to the success of this important scientific undertaking."

The congress thereupon expressed its recognition of the great importance of the projected publication, observed that steps in

the same direction had already been taken by the academies of St. Petersburg and Berlin, and expressed the hope that a report of progress could be presented at the next congress.

The following resolutions, presented respectively by Sections 5 and 6, were adopted:

"Resolved, that a closer union of the historians of the mathematical sciences should be brought about. While such an association, like its objects, should be international, the coöperation of similar and related national societies, journals, museums, etc., should be sought. It is requested that this resolution be placed on the programme of the next congress.

"Resolved, that the Congress greets with warmest sympathy the efforts of mathematicians to secure everywhere the facilities now indispensable for the prosecution of mathematical studies — sufficiently numerous professorial chairs, adequate libraries, drawing-rooms, work-rooms, collections of models, stereopticon apparatus, etc. The Congress expresses the urgent wish that the state and other controlling authorities afford the necessary support for this purpose."

In choosing the place of the next congress the eloquent invitation presented by Professor Volterra on behalf of the Italians for a meeting at Rome in the spring of 1908 was accepted with enthusiasm. A preliminary invitation for 1912 was extended on behalf of the English mathematicians.

Professor Segre, of Turin, delivered an address on "The geometry of to-day and its relations to analysis."

The relations of geometry to analysis depend principally on the fact that in many respects the objects with which the two sciences are occupied — even if under different names — are the same, at least in an abstract sense. The difference between geometry and analysis consists in the problems which they present and still more in the methods by which the problems are treated.

One of the dominant characteristics of the geometry of to-day is the great generality and abstractness which exists in its concepts. This characteristic appears in the higher theory, as also in recent researches on the foundations of geometry. The evolution of this science has relegated to a secondary rank the space intuition which at first constituted one essential element. Reason has become instead the sole essential instrument. Hence it comes that modern geometry can be as rigorous as analysis,

and in fact aspires to be so. Examples of this occur in the determination of the multiplicity of the solutions of geometric problems, and in other questions of enumerative geometry. The latter must resort to algebra to establish its principles, and in return can give algebra many results. The algebraic researches of Kronecker, Hilbert, etc., have great importance for the geometry of algebraic manifolds.

The abstractness already mentioned as characterizing the entities or objects studied in modern geometry extends to the group of transformations which appear as fundamental in such study. Today the dominant geometric tendency, inherited from Riemann, concerns itself with properties invariant under birational transformation. To its triumph much has been contributed by the celebrated memoir of Brill and Noether, as also by the Italian school of geometers. Thus have originated various researches on the algebraic curves, and particularly the more recent ones of Castelnuovo and Enriques on the algebraic surfaces. For birational correspondence and its groups important results have also been reached.

In the direction of increasing generality there may be considered, in place of the algebraic manifold, a more general one to which Segre has devoted a series of papers (about 1890) under the name hyperalgebraic. These arise by assuming algebraic relations, not between the complex coördinates of the geometric elements, but between the real components of those coördinates. For example the so-called Hermite forms, with conjugate complex variables, represent some of the geometric entities considered by Segre, and are studied very easily by these geometric means.

Within the hyperalgebraic class are included as particular cases those composed of the real elements of a given algebraic class. The study of the questions of reality or form is not sufficiently cultivated today; nevertheless important works exist, among which those of Hilbert, Klein, Juel and others may be cited. With this are connected the analysis situs and the researches on the form of the integral curves of differential equations.

There have been considered recently in geometry new species of complex points — the bicomplex, etc., and there have also been employed with advantage new species of complex numbers, for example the numbers $a + b\epsilon$ where $\epsilon^2 = 0$, as used by Study in his Geometrie der Dynamen. On the other hand,

geometric researches have also been made restricting the field of points for example to the points with integral coördinates (Minkowski, Geometrie der Zahlen, etc.), or to those with rational coördinates (Poincaré, *Journal de Mathématiques*, 1901). Thus in this, as in so many other examples, are revealed the bonds uniting geometry more and more closely with analysis — bonds which can give to the one science as to the other immense benefits.

Professor Wirtinger, of Vienna, presented an address on "Riemann's lectures on the hypergeometric series and their significance."

The speaker gave a review of the sources, the plan and the ideas of Riemann's lectures of 1859 on the hypergeometric series. He showed how these ideas came to fruition in subsequent mathematical literature. He recalled first those problems already enunciated by Riemann but today still unsolved. In this connection he formulated other problems leading to new classes of functions.

He dealt in particular with the application of the methods of the principle of Dirichlet and the equation of Fredholm to Riemann's problem in the theory of differential equations, and with the analogy of the relations between the solutions of a non-homogeneous linear differential equation to the solutions of the homogeneous equation on the one hand and to the elliptic integrals and their periods on the other. The classical figure of p parallelograms with $(2p-2)$ branch points he wished treated from points of view similar to those employed for the figure of the single parallelogram and its reduced form. To previous generalizations of the hypergeometric series he added a new one to which he was led through a new and simple representation of the hypergeometric function by means of the elliptic modular functions. The new transcendental functions are definite integrals on the elliptic base extended between $n$ths of periods, and for example for $n = 5$ the group of the icosahedron substitutions has the same significance for them, as the well-known group of the substitutions x, $1/x$, $1-x$, etc., for the ordinary hypergeometric function.

The speaker extended his method also to general algebraic bases and pointed out the relation of his functions to the moduli of the theta functions.

He concluded with the general observation that it is exactly

the working out of what seem at first only interesting particular cases that enables us to make real progress in the general theory. In spite of the deductive form which we give our results, we can as little dispense with the inductive method in mathematics as in any other branch of human learning.

The week ended with further attractive excursions for those whose departure was not too urgent.

The writer acknowledges with cordial appreciation the courteous response of participants in the Congress to his request for abstracts of their papers. By the kind coöperation of Dr. E. B. Wilson, of Yale University, the abstracts of the papers read before the several sections will appear later in the BULLETIN.

H. W. TYLER.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
November, 1904.

THE HEIDELBERG CONGRESS: SECTIONAL MEETINGS.

The several sections of the Congress at Heidelberg met on Tuesday afternoon, August ninth, to organize and arrange programs. For each section there had been previously appointed two or three presidents, of whom one delivered an address, usually merely a few words of welcome and of pleasurable anticipation of the scientific value of the work of the section. In the cases of von Brill and Klein, however, the address widened out into as truly a scientific communication as any presented at the subsequent meetings. Instead of expecting the presidents to take the chair at all the meetings of their sections, the excellent custom of selecting honorary chairmen from among those present was adopted.

The meetings were held Wednesday and Friday mornings. Professor Tyler, representing the AMERICAN MATHEMATICAL SOCIETY, inserted in the *Tageblatt* a statement that the Society would be glad to receive for publication in the BULLETIN abstracts of the various papers, and sent to each speaker whose address could be obtained a circular letter repeating the request. The response was general and cordial, and we have here especially to thank the members of the congress for their

attitude. The length of the abstracts was to be 100 words, but many considerably exceed this limit. The editor, however, has refrained from cutting down the report of the author except in such few instances as have seemed necessary, and possible without injuring the value of the abstract. In one or two cases also editorial exigency has dictated some omission for the sake of avoiding possible error. It is to be regretted that abstracts of some of the more important papers, such as those of Hilbert, on the foundations of arithmetic, and König, on the continuum, have not been received. We may note, however, that the general secretary of the congress, Professor Krazer, is pushing the publication of the proceedings as rapidly as possible, and we may thus hope to have in full all these valuable papers, some of which by the kindness of their authors are here given in brief abstract.

Section I. *Arithmetic and Algebra.*

Presidents: Lüroth and Kneser. Chairmen: Selivanov and Netto.

(1) GORDAN (Erlangen): "Ueber Gleichungen sechsten Grades."†

(2) KÖNIG (Budapest): "Beweis, dass das Continuum keiner wohlgeordneten Menge äquivalent sein kann."

(3) CAPELLI (Naples): "Ein Beitrag zur niederen Zahlentheorie (insbesondere zum Fermat'schen Satze)." †

(4) HOČEVAR (Graz): "Ueber die Bestimmung der linearen Teiler einer algebraischen Form." †

(5) GULDBERG (Christiania): "Ueber lineare Differenzengleichungen." †

(6) MINKOWSKI (Göttingen): "Zur Geometrie der Zahlen."

(7) HILBERT (Göttingen): "Ueber die Grundlagen der Arithmetik."

(8) VORONOÏ (Warsaw): "Sur une propriété du discriminant des fonctions entières."

(9) WIMAN (Upsala): "Die metacyklischen Gleichungen neunten Grades."

(10) LOEWY (Freiburg): "Ueber Gruppen linearer homogener Substitutionen." †

(11) STEPHANOS (Athens): "Sur une catégorie d'équations fonctionnelles." †

† For the convenience of the reader, a dagger is placed after the titles of the communications for which abstracts are given below.

(12) WILSON (New Haven): "On products in additive fields." †

(13) MÜLLER, E. (Constance): "Ueber den Nachlass von E. Schröder." †

1. Gordan established the normal form of the equation of the sixth degree and gave the differential equation which serves to effect its solution.

3. In Fermat's equation $a^{b-1} = 1 + \alpha b$, where a and b are primes, the integer α may be denoted by $\mathrm{E}(a^{b-1}/b)$, if $\mathrm{E}(x)$ denote the integral part of the number x. Capelli gives the expression

$$\alpha = a\,[1 + \mathrm{E}(a^{b-2}/b) + \mathrm{E}(b^{a-2}/a)] - b^{a-2},$$

and a similar expression holds for the generalized equation, $a^{\phi(b)} = 1 + \alpha b$, of Fermat, where a and b are merely relative primes. The proof depends on the relation

$$a\,[a^{\phi(b)-1}]_b + b\,[b^{\phi(a)-1}]_a = 1 + \alpha b$$

where $[x]_a$ denotes the positive residue of an integer x taken modulo a.

4. The linear factors of an arbitrary form, and hence in particular, of one which contains irreducible factors of higher order are determined by means of the theorem: If there be known a special set of values $x_i = a_i$ $(i = 1, 2, \cdots, n)$ for which a linear factor of a form $f(x_1, x_2, \cdots, x_n)$ vanishes and all the remaining factors are different from zero, then the linear factor is determined by $\Sigma_{i=1}^{i=n} x_i\,(\partial f/\partial x_i)_0$, where the index 0 denotes that the variables x are replaced by the values a. If then there be substituted in the equation $f = 0$ the values $x_2 = a_2$, $x_3 = a_3$, $\cdots$, $x_n = a_n$, which are arbitrary save for satisfying the sole condition that the corresponding values $x_1 = a_1^{(1)}, x_1 = a^{(2)}, \cdots, x_1 = a^{(m)}$ are distinct, each linear factor of f is identical with one of the expressions $\Sigma_{i=1}^{i=n} x_i\,(\partial f/\partial x_i)_\lambda$, where the index λ denotes that the values $x_1, x_2, \cdots, x_n$ have been replaced by $a_1^{(\lambda)}, a_2, \cdots, a_n$. It must be stipulated that the form f has no multiple factors, or if it has, that these have been removed by Euclid's algorithm. A modification of the method indicated in this theorem for determining the linear factors is based on the theorem: The greatest common divisor of a form and all

three-rowed minors of its hessian determinant is the product of the linear factors of the form. If this product has been found, one has merely to resolve it into its linear factors.

5. Guldberg treats the linear homogeneous difference equation

$$y_{x+n} + p_x^{(1)} y_{x+n-1} + p_x^{(2)} y_{x+n-2} + \cdots + p_x^{(n)} y_x = 0,$$

where the p_x's denote given functions of x. He shows that almost all formal theories of linear homogeneous differential equations have their analogues in the theory of such difference equations — for example, the theory of the invariant function, of the transformed equation, of the associated equation, and in particular the fact that the integration of the equation is dominated by the linear homogeneous group of transformations which belongs to the equation.

10. After a short account of his researches on reducibility (published in the *Transactions*), Loewy introduces with Burnside the conception of the complete reducibility of a group of linear homogeneous substitutions. Every such group which transforms a definite Hermite form into itself, is completely reducible. Although the complete reducibility of any *finite* group of linear homogeneous substitutions follows from the theorem, not all *infinite* groups of this type are completely reducible. But by them is uniquely determined in order a series of completely reducible groups in which the total number of variables is equal to the number of variables of the original group. These investigations stand in the closest relation to Frobenius's definition of reducibility of linear homogeneous differential equations, if one introduces the conception of the reducibility of such differential equations as fundamental. This conception is particularly fundamental for considering the totality of irreducible linear homogeneous differential equations whose integrals satisfy a given equation of this type.

11. Stephanos begins with the following fundamental proposition: The necessary and sufficient condition that a function $F(x, y)$ of two variables be expressible by the formula

$$F(x, y) = \Sigma \rho_i(x) x_i(y) \qquad (i = 1, 2, \cdots, m)$$

is the identical vanishing of the determinant

$$\begin{vmatrix} F & \frac{\partial F}{\partial x} & \cdots & \frac{\partial^m F}{\partial x^m} \\ \frac{\partial F}{\partial y} & \frac{\partial^2 F}{\partial x \partial y} & \cdots & \frac{\partial^{m+1} F}{\partial x^m \partial y} \\ \vdots & & & \\ \frac{\partial^m F}{\partial y^m} & \frac{\partial^{m+1} F}{\partial x \partial y^m} & \cdots & \frac{\partial^{2m} F}{\partial x^m \partial y^m} \end{vmatrix}.$$

He applies this proposition to discuss the following questions: 1° what functions $f(x)$ give rise to an addition formula $f(x+y) = \Sigma \phi_i(x) f_i(y)$, 2° what functions give rise to a formula $[f(x) - f(y)]/(x-y) = \Sigma \phi_i(x) \psi_i(y)$, 3° what functions yield $[f(x) - f(y)]/(x-y) = \Sigma \phi_i(x) f_i(y)(x-y)^{i-1}$, 4° when does the development of $[f(x) - f(y)]/(x-y)$ by Lagrange's formula lead to a specified result? The full exposition of these matters will be found in the *Rendiconti di Palermo*, volume 18 (1904).

12. Wilson undertook to give some idea of the views of J. Willard Gibbs on multiple algebra and its connection with vector analysis. It was shown that neither the views and methods of Grassmann nor those of Benjamin Peirce were in themselves sufficiently broad to cover the field. Adopting the more general point of view of Gibbs would, in large measure, allay the present strife between different schools of vector analysis. Above all, it is wrong to infer that the vector analysis represents in any adequate way the totality of Gibbs's work on multiple algebra.

13. Müller reported that in continuation of the Algebra der Logik, of which Schröder had published a first volume and the first parts of a second and third volume before his death, there is shortly to appear the second part of volume 2. The theory of relations, as announced in the volumes which have already appeared, has been left to the second part of the third volume. When this may appear is not yet determined, but first will be published a short account of the algebra of logic as set forth in the first two volumes. This will also appear in the near future.

Section II. *Analysis.*

Presidents: Schwarz and Hilbert. Chairmen: Mittag-Leffler, Lindelöf, Hadamard, and Levi-Civita.

(1) SCHLESINGER (Klausenburg): "Ueber das Riemannsche Problem zur Theorie der linearen Differentialgleichungen und daran anschliessende neuere Arbeiten."

(2) BOREL (Paris): "Sur l'approximation des fonctions continues par des polynômes." †

(3) HILBERT (Göttingen): "Ueber Integralgleichungen."

(4) VORONOÏ (Warsaw): "Sur le développement, à l'aide des fonctions cylindriques, des sommes doubles $\Sigma f(pm^2 + 2qmn + rn^2)$, où $pm^2 + 2qmn + rn^2$ est une forme quadratique positive à coefficients entiers."

(5) FRICKE (Braunschweig): "Ueber die Existenz der polymorphen Funktionen auf Riemannschen Flächen." †

(6) BOUTROUX (Paris): "Sur les fonctions entières d'ordre entier." †

(7) MITTAG-LEFFLER (Stockholm): "Sur une classe de fonctions entières."

(8) HADAMARD (Paris): "Sur les équations linéaires aux dérivées partielles." †

(9) CAPELLI (Naples): "Ueber die Additionsformeln der Thetafunktionen." †

2. In this communication Borel takes up his researches on the interpolation of continuous functions by means of polynomials. After recognizing that in certain cases Lagrange's formula for interpolation gives a divergent result, he shows how a formula of interpolation convergent for all continuous functions may be obtained. This result has been developed by the author in his Leçons sur les fonctions de variables réelles, Paris, Gauthier-Villars, November, 1904.

5. Given a closed Riemann surface over the plane of the complex variable; upon it consider a function $f(z)$ which, except in n points $z = e_1, e_2, \cdots, e_n$, has the character of a rational function, and undergoes a linear transformation on passing around these points and on describing the periods. Such a function is called linear-polymorphic or briefly polymorphic. In particular are treated such polymorphic functions as map the surfaces (when suitably cut) upon polygons whose sides are fundamental circles, as in the theory of automorphic functions. The theorems of existence given about twenty years ago by Klein and Poincaré were proved by means of considerations of continuity. Yet in many ways these considerations appear incomplete. Fricke spoke on his researches carried out last

year for the purpose of giving a new foundation and development to those considerations of continuity.

6. It is known that the *order* of an infinite product $G(z)$ having the zeros $a_1, a_2, \cdots$ is the smallest number ρ such that the series $\Sigma a_i^{-\rho-\epsilon}$ and $\Sigma a_i^{-\rho+\epsilon}$ shall be respectively convergent and divergent for all values of ϵ. Boutroux seeks to show that, in case ρ is an integer (the case that is left aside as exceptional in the general theory of integral functions), the behavior and the properties of $G(z)$ are similar to those of the exponential function. For example, let $|a_i|$ be proportional to $(i \log i)^{1/\rho}$. The modulus of $G(z)$ is in general comparable to $e^{A z^\rho}$, A being a positive function of $|z|$. If there be added to $G(z)$ an integral function $g(z)$ of lower order, the zeros of $G + g$ converge toward p straight lines issuing from the origin and spaced at equal angles.

8. The object of this communication by Hadamard is to show that in the case of equations with real characteristics (the hyperbolic case), as in the case of imaginary characteristics, the method of integration should take as its base the *fundamental solution* (a solution singular at a point and on the characteristic conoid issuing from that point, as is $1/r$ for the equation of Laplace). This solution is the only one which is related in any essential way to the question proposed. The reason that those who have occupied themselves with this question have not been able to generalize in a perfectly satisfactory manner the method of Volterra, is because they started with other solutions, solutions containing parasitic singularities. As to the difficulties which at first sight would seem to nullify the method (the introduction of meaningless integrals), they are merely apparent and may be entirely done away with by properly defining the *finite* part of an infinite integral. The communication ends with a property of Bessel functions and hypergeometric functions — a property which it would be interesting to generalize.

9. In treating the elliptic functions it is very useful to have the various particular formulas for the addition of the ϑ-functions of a single variable combined into one formula with arbitrary characteristics. Hermite in 1858 found such a formula having four terms and restricted to integral characteristics, Betti in 1860 and H. Smith in 1866 gave similar expressions.

Capelli now has reduced this general expression to two terms. If

$$\vartheta_{\gamma,g}(u)=\sum_{n=-\infty}^{n=+\infty}e^{\pi i\omega\left(n+\frac{\gamma}{2}\right)^2+2\pi i\left(n+\frac{\gamma}{2}\right)\left(u+\frac{g}{2}\right)}$$

then

$$\vartheta_{\gamma_1,g_1}(u+v)\cdot\vartheta_{\gamma_2,g_2}(u-v)\cdot\vartheta_{\sigma,s}(0)\cdot\vartheta_{\rho,r}(0)$$
$$=\vartheta_{\gamma_1,g_1}(v)\cdot\vartheta_{\gamma_2,g_2}(-v)\cdot\vartheta_{\sigma,s}(u)\cdot\vartheta_{\rho,r}(u)+(-1)^{\sigma}\vartheta_{\gamma_1+1-\sigma,g_1+1-s}(v)\cdot$$
$$\vartheta_{\gamma_2+1-\sigma,g_2+1-s}(-v)\cdot\vartheta_{11}(u)\cdot\vartheta_{\gamma_1+\gamma_2-1,g_1+g_2-1}(u),$$

where $\rho=\gamma_1+\gamma_2-\sigma$ and $r=g_1+g_2-s$. This formula holds for all real and imaginary values of γ_1, γ_2, g_1, g_2. Only the characteristics σ and s need be integers.

Section III. *Geometry.*

Presidents: Von Brill, Fr. Meyer, and Schur. Chairmen: Zeuthen, Segre, Morley, and Guichard.

(1) Von Brill (Tübingen): "Ueber Elimination und Geometrie seit 50 Jahren." †

(2) Macaulay (London): "Intersection of plane curves." †

(3) Guichard (Clermont-Ferrand): "Sur les systèmes triples orthogonaux."

(4) Study (Bonn): "Kürzeste Wege im komplexen Gebiete." †

(5) Meyer (Königsberg): "Grundzüge einer Theorie des Tetraeders."

(6) Rohn (Dresden): "Ueber algebraische Raumkurven." †

(7) Scheffers (Darmstadt): "Ueber Isogonalkurven, Aequitangentialkurven und komplexe Zahlen." †

(8) Schoenflies (Königsberg): "Struktur der perfekten Mengen." †

(9) Zindler (Innsbruck): "Zur Differentialgeometrie der Liniengeometrie." †

(10) Wilczynski (Berkeley): "The general projective theory of space curves and ruled surfaces." †

(11) Andrade (Besançon): "Les mouvements des solides aux trajectoires sphériques."

(12) Knoblauch (Berlin): "Grundformeln der Theorie der Strahlensysteme." †

(13) Lilienthal (Münster): "Ueber äquidistante Kurven." †

(14) AUTONNE (Lyons): "Sur les substitutions crémoniennes dans l'espace à plusieurs dimensions." †

(15) GENESE (Aberystwith): "On some useful theorems in continued multiplication of the regressive product in the real four point space." †

(16) STUDY (Bonn): "Ueber das Prinzip der Erhaltung der Zahl." †

(20) CABREIRA (Lisbon): "Note sur les rapports polygonaux" (by title).

(21) TIKHOMANDRITZKY (Suida): "Die Winkelsumme eines ebenen Dreiecks" (by title).

(22) GIAMBELLI (Turin): "Sul principio della conservazione del numero" (by title).

1. Numerous geometric problems deal with restricted sets of equations for which all the determinants of a matrix vanish; others lead to the condition expressing the fact that a system of equations which are identically satisfied by one or more sets of values of the variables have also in common another solution. Everything that occurs in elimination is capable of treatment by the aid of a method invented by Kronecker and yielding as many solutions of all kinds as are common to a system of equations. By this method also are to be stated the cases in which all the boundaries within which the principle of conservation of number can be applied without danger — and of this Brill's report gave some examples.

2. The equations which must be identically satisfied by the coefficients of a non-homogeneous polynomial M in n variables, in order that M may be divisible by the modular system M_1, M_2, $\cdots$, M_n, are called the linear equations of the manifold (M), which is the intersection of the n manifolds $M_1 = M_2 = \cdots = M_n = 0$. Every proper manifold forming a part of (M) has also its corresponding linear equations. Any such proper manifold which is an isolated point of any degree of multiplicity is called a base point. The paper summarizes the known properties of the linear equations of the intersections of plane curves, and makes some extensions to space of n dimensions.

4. This paper of Study's is concerned with an extension to the imaginary domain of the notion of shortest distance. A method of measurement was developed in which Cayley's

quadratic form is replaced by a Hermite form. The "distance" of two imaginary points in the so-called elliptic case, where the Hermite form is definite, is a real continuum, and can therefore be connected with the notions of "larger" and "smaller." Geodesics can be defined. They lie in straight lines and are identical with a special kind of von Staudt chains. If the coefficients of the Hermite form are real, the geometry based on these notions coincides with elliptic geometry, while in the imaginary domain it has other properties. If n is the number of homogeneous variables, the geometry belongs to a certain group of "motions" and of "transformations of symmetry," the continuous subgroup of which depends upon n^2-1 parameters and is primitive and simple. The whole of this geometry can be interpreted on a certain manifold of $2n-2$ dimensions, which is situated in a euclidean space of n^2-1 dimensions. The imaginary points of a straight line are represented by real points of spheres, all of which have the same radius. Applications were made to the definition of fundamental regions of automorphic functions of $n-1$ variables, and to the representation of all finite projective groups by means of euclidean motions. An abstract of this rather extensive theory will appear in the *Mathematische Annalen*.

6. The problem of determining all kinds of curves R_n^p, of degree n and deficiency p in space has been treated by Noether, Halphen, Valentiner; but we are still far from a complete insight into the relations which arise. Rohn finds that it is well to study the sets of points in which planes and surfaces of the second, third, up to the ith order cut the curve. If these sets are not special, the solution of the problem follows immediately. The solutions may also be brought into a definite form, if the sets are complete special sets. But if they be partial sets, essential difficulties arise and the residual curve must be studied. To every such partial set on R_n^p corresponds an analogous partial set on the residual curve and this is in general no longer special. It may therefore be determined and by it the corresponding set on the curve R_n^p.

7. Scheffers sets over against the conformal group in the plane another group consisting of the contact transformations which carry straight lines into straight lines and segments into segments of equal length. This group plays the same rôle for equitangential curves that the conformal group does for isogonal

curves. As the conformal group is given by the equation $x' + iy' = f(x + iy)$, so this new group can be represented by the equation $u' + jv' = f(u + jv)$. But the system of complex numbers founded on the units $(1, j)$ is not the ordinary system for which $i^2 = -1$, but the other system in which $j^2 = 0$.

8. Schoenflies made a report on his memoir in the *Mathematische Annalen*, volume 59.

9. Zindler defines geometrically the direction of advance in a line space and determines it by coördinates. The rays neighboring to a fixed ray s can be arranged into ∞^3 directions of line space. Of these only ∞^2 are contained in a given complex to which the ray s belongs. They may be represented by ∞^2 suitably chosen neighboring rays of the complex, so that there is obtained a visual representation of the division of the rays of a complex in the neighborhood of s.

10. Wilczynski gave a brief outline of his investigations during the last few years. These investigations belong to a domain which hitherto has been practically untouched. Projective geometry has, on the whole, confined itself to the consideration of algebraic configurations. The general theory of surfaces, congruences, etc., on the other hand, has not been projective, but metrical. The theory of ruled surfaces and of curves in space as developed by Wilczynski is characterized by the fact that it is a projective theory, applicable not only to algebraic cases but in general. It represents the first chapters of a general projective geometry. The work has all appeared or is shortly to appear in the *Transactions*.

12. The formulas by means of which Bianchi proved Guichard's theorems on the deformation of quadric surfaces and which he has taken up in the new edition of his Geometria differenziale depend upon a particular assumption concerning the parameters of a system of rays and contain two sorts of quantities — those which may be made independent of the choice of the parameters and those which cannot. These disadvantages may be alleviated by making fundamental, not differentiation with respect to the parameters, but those operations which are connected with the theory of a pair of quadratic differential forms and which possess the property of invariance for the system of rays or for its defining surface (Ausgangsfläche). Knoblauch gave an example of the difference between the for-

mulas that arise in this way and those obtained by Bianchi. He reserved the development of the formulas themselves for another place.

13. On pseudospherical surfaces, and only on them, do the asymptotic lines form an equidistant system. In a corresponding manner one may ask if there are surfaces of positive curvature on which the characteristic curves (Encyklopädie, volume 3, part 3, page 115) form an equidistant system. The surfaces in question must be surfaces of translation. In the integration of the five differential equations which arise one must distinguish between Weingarten and non-Weingarten surfaces. For the latter the theorem of existence follows from the actual integration and it is seen that the surfaces in question are isothermal.

14. Autonne regards $2N$ variables x_i and u_i $(i = 1, 2, \cdots, N)$ connected by three relations $\Sigma\, xu = 0$, $\Sigma\, cx = \Sigma\, gu = 1$, ($c_i$, g_i arbitrary numerical constants), as the coördinates of an element (x, u) in a space of $N - 1$ dimensions. The element (x, u) is made up of the point x and the plane u, which have the x_i's and the u_i's as their respective homogeneous coördinates. The plane u passes through the point x. Let the expression $f(x^m; u^{m'})$ denote a polynomial homogeneous in the x's and in the u's, the degrees of homogeneity being respectively m and m'. The transformation

$$G = \begin{vmatrix} x_i & \phi_i(x^m;\ u^{m'}) \\ u_i & \psi_i(x^n;\ u^{n'}) \end{vmatrix}$$

will be of Cremona type, if it is 1° birational with respect to the coördinates of an element and 2° a transformation of contact. The general properties of Cremona transformations and the basis of their classification is studied.

15. If BC be a regressive product and if A be chosen so that AC is a scalar, then 1) if A be contained in B, $A.\ BC = AC.\ B$; 2) if A be not contained in B, $A.\ BC = AC.\ B - AB.\ C$. Theorem 1 was first derived from one of Grassmann's, and then proved geometrically for real space, and its uses shown. Theorem 2 was observed to be true also when BC is not regressive in all cases except one, namely when A, B, C each represents a line.

16. In this communication Study makes some criticisms on the so-called principle of conservation of number, as stated by Schubert in his "Kalkul der abzählenden Geometrie." The paper will appear in the *Archiv für Mathematik*.

E. B. WILSON.

(*To be continued.*)

NOTES.

THE Annual Register of the AMERICAN MATHEMATICAL SOCIETY for the year 1905 is now in press and will soon be issued. The Register contains this year a complete catalogue of the Society's library, which now amounts to over 2000 volumes. The membership of the Society is now 473, including 32 life-members. The treasurer's report shows a balance of $ 3884.28 on hand December 27, 1904.

THE presidential address "On mathematical progress in America," delivered by Professor THOMAS S. FISKE at the annual meeting of the Society will be published in the February BULLETIN.

THE annual meeting of the Association of teachers of mathematics in the Middle States and Maryland was held at Princeton, New Jersey, November 26. It was attended by about 100 teachers representing nearly every college and secondary school within its territory. Sixteen new members were admitted, thereby increasing the membership to 297.

The following officers were elected: President, Professor T. S. FISKE; Vice-President, Dean H. B. FINE; Secretary, Dr. ARTHUR SCHULTZE. President Wilson of Princeton University, delivered the address of welcome. The following papers were read: "On defining the various forms of numbers ordinally," by H. B. FINE; "The relative importance of the topics in elementary algebra," by A. HEIKES; "Some of the conditions under which mathematical teachers are working," by M. CURTIS; "The great movements now taking place in the teaching of mathematics in other countries," by G. LEGRAS.

THE Central association of science and mathematics teachers held its fourth meeting at the Northwestern University Profes-

sional School in Chicago, November 25–26. President HOLGATE gave an address of welcome. At the opening general session an address was given by Professor D. C. JACKSON: "Desirable product from the teacher of mathematics; the point of view of an engineering teacher." The following papers were read before the mathematical section: "Some problems confronting the teacher of geometry," by C. W. SUTTON; "The high school mathematics club," by C. W. NEWHALL; "The introduction of meteorology into the course of instruction in mathematics and physics," by C. ABBE; "Progress in correlation of physics and mathematics," by F. S. BISHOP. The officers of the section for the coming year are: Chairman, H. E. COBB; Vice-Chairman, J. V. COLLINS; Secretary, Miss MABEL SYKES.

ON Saturday, November 12, 1904, a conference of the teachers of mathematics in schools affiliated or coöperating with the University of Chicago was held at the university under the chairmanship of Professor E. H. MOORE. The paper of the conference was by Miss MABEL SYKES, of the South Chicago High School, on "Conservative and radical elements in the teaching of mathematics." It was followed by an animated discussion relative mainly to the desirability of closer interrelation (in accordance with established and current usage in Germany, France and England) of algebra, geometry and physics, both in time and order of teaching and in matter taught. Especial interest attached to the reports on various experiments in progress. The *School Review* for January, 1905 will give a full digest of the paper and the discussion.

AT the annual meeting of the London mathematical society held on November 10, the following officers were elected: A. R. FORSYTH, president; A. E. H. LOVE and J. H. GRACE, secretaries; J. LARMOR, treasurer; three vice-presidents and nine other members of the council. The following papers were read: By Professor H. LAMB (president's address), "Theory of waves on liquids"; by Professor V. VOLTERRA, "Note on the application of the method of images to problems in vibration"; by Mr. G. H. HARDY, "Zeros of certain classes of integral Taylor's series"; by Rev. E. W. BARNES, "Linear differential equations of the first order"; by Mr. H. HILTON, "Curves on a conicoid"; by Dr. H. F. BAKER, "Remarks on

alternants and continuous groups"; by Dr. J. W. L. GLAISHER, "On the expansion of the elliptic and zeta functions of $2/3\ k$ in powers of q"; by Mr. J. E. WRIGHT, "Examples of perpetuants"; by Professor R. W. GENESE, "Two simple results in the attraction of uniform wires obtained by quaternions, with, for comparison, their verification by the geometry of the complex"; by Mr. P. W. WOOD, "On the reducibility of covariants of binary quantics of infinite order"; by Professor W. BURNSIDE, "On some properties of groups of odd order."

AT the Breslau meeting of German scientists and physicians, a committee was appointed to formulate a plan by which a larger part of the instruction given in the gymnasia could be devoted to mathematics and science than is now the case. Professor A. GUTZMER, as chairman of the association, and Professor F. KLEIN, as representative of the German mathematical society, are among its members.

THE Deutsche Mathematiker-Vereinigung now numbers 635 members. Professor E. PAPPERITZ has been elected to the council. The next regular meeting will be held in Meran, in the Tyrol, in September, 1905, in connection with the seventy-eighth general meeting of German naturalists and physicians.

AMONG the books just issued or about to appear from the publishing house of B. G. Teubner in Leipzig, may be mentioned: "Theorie der Elektrizität, Band 1, Einführung in die Maxwellsche Theorie der Elektrizität, von Dr. A. FÖPPL; zweite umgearbeitete Auflage," by Dr. M. ABRAHAM; Band 2, "Die höheren Probleme der Elektrodynamik," bearbeitet von Dr. M. ABRAHAM; "The Dynamics of particles, and of rigid, elastic and fluid bodies," by ARTHUR GORDON WEBSTER; "Entwickelungen nach oszillierenden Funktionen. Fünfte (Schluss) Lieferung," by H. BURKHARDT; "Wissenschaftliche Abhandlungen über theoretische Physik," in two volumes, by H. A. LORENTZ.

THE firm of Gauthier-Villars, Paris, announces the following works in press: Lessons on discontinuous functions, by R. BAIRE, edited by A. DENJOY; Correspondence of Hermite and Stieltjes, volumes 1 and 2, by B. BAILLAUD and H. BOURGET.

UNIVERSITY OF PARIS. The following courses will be given during the second semester of 1905. — By Professor E.

PICARD: Theory of functions of several variables, two hours. — By Professor E. GOURSAT: Partial differential equations, two hours. — By Professor P. PAINLEVÉ: General laws of motions of systems, analytic mechanics, hydrostatics and hydrodynamics, two hours. — By Professor H. ANDOYER: Program to obtain the certificate in astronomy, two hours. — By Professor J. BOUSSINESQ: Equilibrium of elasticity of the sphere, propagation of motion from a center in an indefinite homogeneous elastic medium, two hours. — By Professor G. KOENIGS: Principles of elasticity, mechanics and resistance of materials, two hours. — By Professors P. APPELL and L. RAFFY: Course in general mathematics, two hours. — By Professor J. TANNERY: Differential and integral calculus, two hours. — The courses of Professors E. BOREL and J. HADAMARD have not been announced.

A MOVEMENT is now on foot to erect a monument at Laibach, Austria, to the memory of GEORG FREIHERR VON VEGA, well known to the mathematical world for his contributions to the practical use of logarithms, his Tables having passed through no less than eighty editions. Contributions are solicited and may be sent to Professor A. Krazer, Westendstrasse 57, Karlsruhe, Germany.

THE Academy of Sciences of St. Petersburg has awarded its Schubert prize for 1905 to Dr. G. W. HILL. The three previous awards of the Schubert Prize were to Gyldèn in 1889, Tisserand in 1893, and Newcomb in 1897.

AMONG the awards made by the Royal society of London during the year 1904, the following are for mathematical investigations: The royal medal to Professor W. BURNSIDE, of Cambridge, for his researches in the theory of groups; the Sylvester medal to Professor G. CANTOR, of Halle, for his researches in the theory of aggregates and of sets of points of the arithmetic continuum, of transfinite numbers and of Fourier's series.

PROFESSOR G. KOWALEWSKI, of Greifswald, and Dr. F. LONDON, of Breslau, have been appointed associate professors of mathematics at the University of Bonn.

PROFESSOR G. LANDSBERG, of Heidelberg, has been appointed associate professor of mathematics at the University of Breslau.

AT the technical school of Darmstadt, Professor S. GUNDELFINGER has recently celebrated the twenty-fifth anniversary of his appointment as professor of mathematics. A new course on the calculus of variations is given by him during the present semester, and Professor G. SCHEFFERS is conducting the course on the calculus, formerly given by Professor Gundelfinger.

Dr. PANEK has been appointed professor of mathematics at the Bohemian technical school at Prague.

DR. M. WEBER, government building inspector, of Berlin, has been appointed professor of mechanics at the technical school of Hanover.

DR. K. TH. VAHLEN, of Königsberg, has been appointed associate professor of mathematics at the University of Greifswald.

PROFESSOR A. KNESER, of the mining academy of Berlin, has been appointed professor of mathematics at the University of Breslau. He will assume the new office April 1, 1905.

DR. J. GRÜNWALD has been appointed docent in mathematics at the technical school of Vienna.

MR. J. H. JEANS, M.A., of Trinity College, has been appointed university lecturer in mathematics at Cambridge University. Appointments to these lectureships, which are four in number, are made for five years.

PROFESSOR R. S. WOODWARD, of Columbia University, has been elected president of the Carnegie Institution.

AT the recent celebration of the one hundred and fiftieth anniversary of the founding of Columbia University, the new professorship of mathematics, recently filled by the appointment of Professor C. J. KEYSER, was named the Adrain professorship, in honor of ROBERT ADRAIN, one of the founders of the theory of observational errors, who held the chair of mathematics in Columbia College from 1813 to 1825.

MR. B. H. CAMP has been appointed instructor in mathematics at Wesleyan University.

DR. G. D. RICHARDSON has been appointed instructor in mathematics at Yale University.

DR. FRANCESCO CHIZZONI, professor of geometry at the University of Modena, died September 23, at the age of 56 years.

THE death is announced, on November 27, 1904, of Professor PAUL TANNERY, the well known French authority on the history of mathematics.

PROFESSOR ACHSAH M. ELY, head of the department of mathematics at Vassar College since 1887, died on December 13. Miss Ely had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1891.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABHANDLUNGEN zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Begründet von M. Cantor. Heft 18: Heiberg (J. L.), Mathematisches zu Aristoteles; Müller (C. H.), Studien zur Geschichte der Mathematik, insbesondere des mathematischen Unterrichts an der Universität Göttingen im 18. Jahrhundert; Lindt (R.), Das Prinzip der virtuellen Geschwindigkeiten, seine Beweise und die Unmöglichkeit seiner Umkehrung bei Verwendung des Begriffes "Gleichgewicht eines Massensystems." Leipzig, Teubner, 1904. 8vo. 196 pp. M. 6.00

——. Heft 19: N. I. Lobatschefskij's Imaginäre Geometrie und Anwendung der imaginären Geometrie auf einige Integrale. Aus dem Russischen übersetzt und mit Anmerkungen herausgegeben von H. Liebmann. Leipzig, Teubner, 1904. 8vo. 11 + 188 pp., 1 plate. M. 8.00

BAILLAUD (B.). See HERMITE.

BOLZA (O.). (1) Concerning the geodesic curvature and the isoperimetric problem on a given surface. (2) Proof of the sufficiency of Jacobi's condition of a permanent sign of the second variation in the so-called isoperimetric problem. Chicago, University of Chicago Press, 1904. 4to. 16 pp. $0.25

BOREL (E.). Leçons sur les fonctions de variables réelles et leur représentation par des séries de polynômes, professées à l'Ecole Normale supérieure et rédigées par M. Fréchet; avec notes par P. Painlevé et H. Lebesgue. (Collection de monographies sur la théories des fonctions.) Paris, Gauthier-Villars, 1904. 8vo. 162 pp. Fr. 4.50

BOURGET (H.). See HERMITE.

BRÜSER (W.). Untersuchungen über die 6-gliedrige halbeinfache Transformationsgruppe. Münster, 1903. 8vo. 31 pp. M. 1.00

CESÀRO (E.). Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung. Mit zahlreichen Uebungsbeispielen. Nach einem Manuskript des Verfassers deutsch herausgegeben von G. Kowalewski. Leipzig, Teubner, 1904. 8vo. 6 + 894 pp. Cloth. M. 15.00

DICKSON (L. E.). (1) Ternary orthogonal groups in a general field. (2) The groups defined for a general field by the rotation group. Chicago, University of Chicago Press, 1904. 4to. 26 pp. $0.50

FABER (G.). Ueber Reihenentwicklungen analytischer Funktionen. (Diss.) München, 1903. 8vo. 67 pp.

FALTER (L.). Die erkenntnistheoretischen Grundlagen der Mathematik bei Kant und Fries. Giessen, 1903. 8vo. 71 pp. M. 2.00

FISHER (I.). Kurze Einleitung in die Differential- und Integralrechnung ("Infinitesimalrechnung"). Aus der durch mehrere Verbesserungen des Verfassers vervollständigten dritten englischen Ausgabe übersezt von N. Pinkus. Leipzig, Teubner, 1904. 8vo. 6 + 72 pp.

FRÉCHET (M.). See BOREL (E.).

FUCHS (L.). Gesammelte mathematische Werke. Herausgegeben von R. Fuchs und L. Schlesinger. Vol. I: Abhandlungen (1858–1875). Redigiert von L. Schlesinger. Berlin, Mayer & Müller, 1904. 8vo. 8 + 476 pp., portrait. M. 30.00

GMEINER (J. A.). See STOLTZ (O.).

GRESSLER (J.). Ueber die Charakteristiken partieller Differentialgleichungen. Viersen, 1904. 4to. 5 pp.

HALLER (S.). Untersuchung der Brennpunktscurve eines Kegelschnittbüschels. Mit besonderer Berücksichtigung der gestaltlichen Verhältnisse. (Diss.) München, 1903. 8vo. 42 pp.

HARTOGS (F.). Beiträge zur elementaren Theorie der Potenzreihen und der eindeutigen analytischen Funktionen zweier Veränderlichen. (Diss.) München, 1904. 8vo. 79 pp.

HEIBERG (J. L.). See ABHANDLUNGEN.

HERMITE et STIELTJES. Correspondance. Publiée par les soins de B. Baillaud et H. Bourget. Avec une préface d'Emile Picard. Tome I: 1882–89. Paris, Gauthier-Villars, 1905. 8vo. 20 + 480 pp. Fr. 16.00

HILB (E.). Beiträge zur Theorie der Lamé'schen Funktionen. (Diss.) München, 1903. 8vo. 60 pp.

KOWALEWSKI (G.). See CESÀRO (E.).

LAGALLY (M.). Ueber Flächen mit sphärischen Krümmungslinien, vom kugelgeometrischen Standpunkt aus betrachtet, und die entsprechenden Flächen des Linienraumes. (Diss.) München, 1903. 8vo. 86 pp.

LEBESGUE (H.). See BOREL (E.).

LECHALAS (G.). Introduction à la géométrie générale. Paris, Gauthier-Villars, 1904. 16mo. 9 + 59 pp. Fr. 1.75

LIEBMANN (H.). See ABHANDLUNGEN.

LINDT (R.), LOBATSCHEFSKIJ (N. I.). See ABHANDLUNGEN.

MASCHKE (H.). Invariants and covariants of quadratic differential quantics of n variables. Chicago, University of Chicago Press, 1904, 4to. 14 pp. $0.25

MILLER (G. A.). On the subgroups of an abelian group. 1904. 4to. (*Annals of Mathematics*, Vol. 6, pp. 1–6.)

MÜLLER (C. H.). See ABHANDLUNGEN.

PAINLEVÉ (P.). See BOREL (E.).

PICARD (E.). See HERMITE.

PINKUS (N.). See FISHER (I.)

SALVERT. Mémoire sur une classe de quadratures de fonctions elliptiques par rapport à leur module. Paris, Gauthier-Villars, 1903. 8vo. 142 pp. Fr. 4.00

SCHLESINGER (L.). See FUCHS (L.).

SPIEGEL (M.). Methode der Integration der linearen Differentialgleichungen zweiter Ordnung mit linearen Koeffizienten durch bestimmte Integrale (Fortsetzung und Schluss). (Progr.) St. Pölten, 1904. 8vo. 26 pp.

STIELTJES. See HERMITE.

STOLZ (O.) und GMEINER (J. A.). Einleitung in die Funktionentheorie. 2te, umgearbeitete und vermehrte Auflage der von den Verfassern in der "Theoretischen Arithmetik" nicht berücksichtigten Abschnitte der "Vorlesungen über allgemeine Arithmetik" von O. Stolz. (In 2 Abteilungen.) Abteilung I. Leipzig, Teubner, 1904. 8vo. 6+242 pp. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. XIV.) M. 6.00

TEMPEL (H.). Die Einführung elliptischer Koordinaten bei den Spezialfällen der Komplexe zweiten Grades. (Diss.) München, 1904. 8vo. 117 pp.

WAGNER (J.). Ueber eine besondere zwei-zweideutige Verwandtschaft. (Diss.) München, 1903. 8vo. 33 pp.

WIESE (C.). Synthetische Untersuchung über Flächen dritter Ordnung mit Doppelpunkten. Münster, 1904. 8vo. 35 pp. M. 1.00

WINKLER (J.). Die einem gegebenen Ellipsoid eingeschriebenen bezw. umgeschriebenen Tetraeder, welche ein Maximum bezw. Minimum des Inhaltes besitzen. (Progr.) Kalksburg, 1904. 8vo. 39 pp.

II. ELEMENTARY MATHEMATICS.

ANDRAE (G.). Mathematische Aufgaben für die Maturitätsprüfung. Mediasch, 1903. 4to. 22 pp.

AUFLÖSUNGEN zur mathematischen Aufgabensammlung von E. Heis. Von einem Praktiker. Quadratische Gleichungen mit einer Unbekannten. München, Haushalter, 1904. 8vo. 104 pp. M.0.50

BACH (H.). See HERCHER (B.).

BARNARD (S.) and CHILD (J. M.). New geometry for senior forms. London, Macmillan, 1904. 8vo. 348 pp. Cloth. 3s. 6d.

BETTINI (B.). See CIAMBERLINI (C.).

BORGMEYER (J.). Methodische Zusammenstellung des planimetrischen Lernstoffs für die mittleren Classen. Meppen, 1904. 8vo. 52 pp.

BURG (R.). Sammlung algebraischer Aufgaben für gewerbliche und technische Lehranstalten, nebst einer Abhandlung über das Stabrechnen. Heft 4: Gleichungen (3 Teil); Proporzionalität; vermischte Aufgaben; Summen; Exponentialgleichungen, geometrische Reihen; Zinseszins. 52 pp. Heft 5: Stabrechnen mit Rechenstab und Uhr. 52 pp. Frankfurt a. M., Auffarth, 1905. 8vo. M. 1.50

CHAILAN (E.). Géométrie, à l'usage des élèves de l'enseignement secondaire; second cycle, divisions A et B. 3e édition. Paris, Poussielgue, 1904. 16mo. 8 + 240 pp.

CHILD (J. M.). See BARNARD (S.).

CIAMBERLINI (C.) e BETTINI (B.). Geometria razionale per il ginnasio superiore. Firenze, Bemporad, 1904. 16mo. 174 pp. (Biblioteca scolastica.) Fr. 1.25

CLASEN (R.). See HERCHER (B.).

CORTASSA (G.). Elementi di algebra per le scuole tecniche. Torino, Vaccarino, 1904. 8vo. 98 pp. Fr. 0.80

DAVIÈS (G.). Mathématiques. 2e édition, revue et très augmentée. Paris, Dunod, 1905. 16mo. 11 + 550 pp.

DIEKMANN (J.). See KOPPE.

DUPORT (J. B.). Lehrbuch der Arithmetik und Geometrie für die IV. und V. Klasse der Mädchen-Lyzeen. Wien, Deuticke, 1904. 8vo. 6 + 249 pp. Cloth. M. 3.00

GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. Zum Gebrauche für Schule und Praxis bearbeitet. Teil II: Fünfstellige logarithmisch-trigonometrische Tafeln für Dezimalteilung des Quadranten. Stereotyp-Druck. 3te Auflage. Halle, Strien, 1904. 8vo. 140 + 18 pp. M. 6.00

GAZZANIGA (P.). Libro di aritmetica generale e di algebra elementare. 4a edizione, notevolmente accresciuta e migliorata. Verona e Padova, Drucker, 1904. 8vo. 322 + 96 pp. Fr. 3.50

GODFREY (C.) and SIDDONS (A. W.). Elementary geometry. Solutions of the exercises by E. A. Price. Cambridge, 1904. 8vo. Cloth 5s.

HANDY BOOK of logarithms. With practical geometrical appendix. London, Blackie, 1904. 8vo. 128 pp. 2s.

HERCHER (B.). Lehrbuch der Arithmetik und Algebra zum Gebrauch an Gymnasien. 2te Auflage. Nach den neuen Lehrplänen bearbeitet von R. Clasen und H. Bach. Leipzig, List, 1904. 8vo. 71 pp. M. 1.30

HERRLINGER (J.). 105 ausführlich gelöste arithmetische Aufgaben aus der II. Dienstprüfung für würtembergische evangelische Volksschullehrer. Stuttgart, Bonz, 1905. 8vo. 6 + 96 pp. M. 1.60

HOLBORN (L.). See LOGARITHMENTAFELN.

HOPKINS (J. W.) and UNDERWOOD (P. H.). A first book of algebra. New York, Macmillan, 1904. 12mo. 9 + 245 pp. Cloth. $0.50

JACOB (J.). Lehrbuch der Arithmetik für Untergymnasien. Abteilung I: Lehrstoff der 1. und 2. Klasse. Wien, Deuticke, 1905. 8vo. 136 pp. Cloth. M. 1.50

KOPPE und DIEKMANN'S Geometrie zum Gebrauche an höheren Unterrichtsanstalten. (20ste Auflage.) Ausgabe für Gymnasien. Teil I: Planimetrie. 4te Auflage der neuen Bearbeitung von J. Diekmann. Essen, Baedeker, 1904. 8vo. 4 + 208 pp., 8 plates. Cloth. M. 2.40

KUTNEWSKY (M.). See MÜLLER (H.).

LIGHTFOOT (J.). Algebraic factors made easy. Key. London, Simpson, 1904. 8vo. 1s.

LOGARITHMENTAFELN, vier- und fünfstellige, nebst einigen physikalischen Konstanten. Aufgestellt von L. Holborn und K. Scheel. Braunschweig, Vieweg, 1904. 8vo. 24 pp. M. 0.80

MARTUS (H. C. E.). Mathematische Aufgaben zum Gebrauche in den obersten Klassen höherer Lehranstalten. Aus den bei Reifeprüfungen an deutschen höheren Schulen gestellten Aufgaben ausgewählt und mit Hinzufügung der Ergebnisse (IV. Teil) zu einem Uebungsbuche vereint. Teil III: Aufgaben. 2te Doppelauflage, mit vermehrten Aufgaben. Dresden, Koch, 1904. 8vo. 8 + 180 pp. M. 4.20

MAYO (C. H. P.). Elementary trigonometry. London, Longmans, 1904. 8vo. 220 pp. Cloth. 3s. 6d.

MÜLLER (H.) und KUTNEWSKY (M.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. (H. Müller's mathematisches Unterrichtswerk.) Ausgabe A, für Gymnasien. Teil 2. 2te, verbesserte und stark gekürzte Auflage. Leipzig, Teubner, 1905. 8vo. 8 + 273 pp. Cloth. M. 2.20

NASSÒ (M.). Aritmetica generale ed algebra ad uso dei licei, secondo il programma governativo del 24 ottobre 1900, con copiose note storiche, molti consigli pratici per indirizzare l'alunno alla risoluzione degli esercizi, più di 2200 exercizi e problemi graduali da risolvere e circa 400 esercizi e problemi minutamente risolti. 3a edizione. Torino, Tipografia Salesiana, 1904. 8vo. 492 pp. Fr. 3.50

NITSCHE (J.). Lehr- und Uebungsbuch der Arithmetik für die III. und IV. Gymnasialklasse. Wien, Deuticke, 1904. 8vo. 3 + 138 pp. Cloth. M. 1.80

PRESSLAND (A. J.) and TWEEDIE (C.). Elementary trigonometry. Part 1. Revised edition. London, Oliver, 1904. 8vo. 224 pp. Cloth. 2s. 6d.

PRICE (E. A.). See GODFREY (C.).

REINHARD (A. I.). A high school algebra. New York, American Book Co., 1904. 12mo. 7 + 422 pp. Cloth. $1.00

REUSCH (J.). Planimetrische Konstruktionen in geometrographischer Ausführung. (Progr., Thann.) Leipzig, Teubner, 1904. 8vo. 13 + 84 pp. M. 1.00

SCHEEL (K.). See LOGARITHMENTAFELN.

SCHMITT (P.). Die Behandlung der Raumlehre in der Volksschule. Langensalza, Schulbuchhandlung, 1904. 8vo. 4 + 151 pp. M. 1.50

SCHRÖN (L.). Siebenstellige gemeine Logarithmen der Zahlen von 1 bis 10800. 25ste revidirte Ausgabe. Braunschweig, 1904. 8vo. 226 pp. M. 2.40

SIDDONS (A. W.). See GODFREY (C.).

SMITH (C.). Solutions of problems and theorems in Charles Smith's Theoretical Problems. London, Macmillan, 1904. 8vo. Cloth. 6s.

SPRAGUE (R. H.). Geometry for technical students. Introduction to pure and applied geometry and the mensuration of surfaces and solids. London, 1904. 8vo. Cloth. 1s. 3d.

TWEEDIE (C.). See PRESSLAND (A. J.).

III. APPLIED MATHEMATICS.

ABRAHAM (M.). Theorie der Elektrizität. Erster Band: Einführung in die Maxwellsche Theorie der Elektrizität von Dr. A. Föppl. Leipzig, Teubner, 1904. 8vo. 18 + 443 pp. Cloth. M. 12.00

BÉNARD (H.). See BOLTZMANN (L.).

BENOIT (C.). See MARX (A.).

BOLTZMANN (L.). Leçons sur la théorie des gaz; traduites par A. Galotti et H. Bénard. Avec une introduction et des notes de M. Brillouin. 2e partie. Paris, Gauthier-Villars, 1905. 8vo. 7 + 280 pp. Fr. 10.00

BOULAD (F.). Note sur un tracé géométrique des paraboles cubiques et ses applications aux lignes d'influence dans les parties continues. Paris, Bernard, 1904. 8vo. Fr. 2.00

BOUSSINESQ (J.). Complément au mémoire intitulé: Recherches théoriques sur l'écoulement des nappes d'eau infiltrées dans le sol et sur le débit des sources. Paris, Gauthier-Villars, 1904. 4to. 32 pp. Fr. 2.00

BRILLOUIN (M.). See BOLTZMANN (L.).

BUCHERER (A. H.). Mathematische Einführung in die Elektronentheorie. Leipzig, Teubner, 1904. 8vo. 4 + 148 pp.

FELDMANN (C.). See HERZOG (J.).

FÖPPL (A.). See ABRAHAM (M.).

FUHRMANN (A.). Aufgaben aus der analytischen Mechanik. Uebungsbuch und Literaturnachweis für Studierende der Mathematik, Physik, Technik u. s. w. (In 2 Teilen.) Teil I: Aufgaben aus der analytischen Statik fester Körper. 3te, verbesserte und vermehrte Auflage. Leipzig, Teubner, 1904. 8vo. 12 + 206 pp. Cloth. M. 3.60

GALOTTI (A.). See BOLTZMANN (L.).

GUIDI (C.). L'ellisse di elasticità nella scienza delle costruzioni; appendice al Corso di scienze delle costruzioni, dell'anno 1903–1904. Torino, Bertolero, 1904. 8vo. 24 pp.

HART (G.). Les turbines à vapeur. Paris, Gauthier-Villars, 1904. 8vo. 139 pp., 1 plate. Fr. 4.00

HEMMING (G. W.). Billiards mathematically treated. London, Macmillan, 1904. 2d edition. 8vo. 3s. 6d.

HERZOG (J.) und FELDMANN (C.). Die Berechnung elektrischer Leitungsnetze in Theorie und Praxis. 2te Auflage, in 2 Teilen. Teil 2: Dimensionierung der Leitungen. Berlin, Springer, 1905. 8vo. 8 + 454 pp. Cloth. M. 12.00

KLAUS (A.). Ueber die Entwicklung der kinetischen Gastheorie und ihre Bedeutung für die moderne Physik. (Diss.) Freiburg, 1904. 8vo. 91 pp.

LEONARD (J. H.) and SALMON (W. H.). A further course of practical science, mechanics, hydrostatics, and heat. New York, Dutton, 1904. 12mo. 224 pp. Cloth. $0.60

MACH (E.). Die Mechanik in ihrer Entwickelung, historisch-kritisch dargestellt. 5te verbesserte und vermehrte Auflage. Leipzig, Brockhaus, 1904. 8vo. 16 + 561 pp. (Internationale wissenschaftliche Bibliothek, Vol. 59.) M. 8.00

MARX (A.). L'ether principe universel des forces. Resumé par C. Benoit. Paris, Gauthier-Villars, 1905. 8vo. 10 + 217 pp. Fr. 6.50

MICHELSON (A. A.). Light waves and their uses. Chicago, University of Chicago Press, 1904. 8vo. 174 pp., 3 plates. Cloth. $2.00

MOTTEZ (L.). La matière, l'ether et les forces physiques. Paris, Gauthier-Villars, 1904. 8vo. 236 pp. Fr. 4.00

NEUDECK (G.). Die Dampfturbine. Kiel, Universitätsbuchhandlung, 1904. 8vo. 5 + 89 pp. M. 2.80

ROUBAUDI (C.). Cours de géométrie descriptive pour l'enseignement secondaire à l'usage des élèves de la classe de mathématiques (sections A et B), des candidats au baccalauréat, à l'Institut agronomique, à l'Ecole navale et à l'Ecole de Saint-Cyr. 2e édition, refondue et conforme aux programmes du 31 mai 1902. Paris, Masson, 1904. 8vo. Fr. 5.00

SALMON (W. H.). See LEONARD (J. H.).

SOUCHON (A.). La construction des cadrans solaires, ses principes, sa pratique, précédée d'une histoire de la gnomonique. Paris, Gauthier-Villars, 1905. 8vo. 8 + 52 pp., 2 plates. Fr. 2.75

SPARRE. Sur le mouvement des projectiles oblongs autour de leur centre de gravité. Paris, Gauthier-Villars, 1904. 8vo. 36 pp. Fr. 1.00

STENZL (F.). Einfluss der Erddrehung auf Bewegungen an der Erdoberfläche in neuer Darstellung. (Progr.) Zwittau, 1904. 8vo. 22 pp.

SUSLOV (G.). Theory of the potential and hydrodynamics. Vol. I: Theory of the potential. Kiev, 1904. 8vo. 6 + 167 pp. (Russian.)

TOLLE (M.). Die Regelung der Kraftmaschinen. Berechnung und Konstruktion der Schwungräder, des Massenausgleichs und der Kraftmaschinenregler in elementarer Behandlung. Berlin, Springer, 1905. 8vo. 11 + 461 pp., 9 plates. Cloth. M. 14.00

THOMSON (J. J.). Elements of the mathematical theory of electricity and magnetism. London, Clay, 1904. 3d edition. 8vo. 552 pp. Cloth. 10s.

ULLER (K.). Beiträge zur Theorie der elektromagnetischen Strahlung. (Diss.) Rostock, 1903. 4to. 42 pp.

WALKER (J.). Analytical theory of light. London, Clay, 1904. 8vo. 432 pp. Cloth. 15s.

WALLENTIN (I.). Einleitung in die theoretische Elektrizitätslehre. Leipzig, Teubner, 1904. 8vo. 10 + 444 pp. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. XV.) M. 12.00

WEBSTER (A. G.). The dynamics of particles, and of rigid, elastic, and fluid bodies, being lectures on mathematical physics. Leipzig, Teubner; (New York, Stechert;) 1904. 8vo. 12 + 588 pp. Cloth. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften, mit Einschluss ihrer Anwendungen, Vol. XI.) M. 14.00

WEYRAUCH (J. J.). Grundriss der Wärmetheorie. Mit zahlreichen Beispielen und Anwendungen. Nach Vorträgen an der k. technischen Hochschule in Stuttgart. Erste Hälfte: I. Erhaltung der Energie; 1. Hauptsatz.—II. Wärme und Arbeit; 2. Hauptsatz.—III. Ueber Wärmemotoren im allgemeinen.—IV. Von den Gasen.—V. Ueber Luftmaschinen.—VI. Aus der Chemie und kinetischen Gastheorie.—VII. Ueber Verbrennungsmotoren. Stuttgart, Wittwer, 1905. 8vo. 15 + 324 pp. M. 12.00